Asset Pricing Review Session 1

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Outlines for review sessions

First Part (Static Asset Pricing)

1. Choice under uncertainty
2. Static portfolio choice
3. Static asset pricing
4. Stochastic discount factor

Second Part (Intertemporal Asset Pricing)

1. Present value relations
2. Long run risk (BY)
3. Intertemporal CAPM (CV)
4. Rare Disaster (Martin)
5. Stochastic volatility (BKY and CGPT)
6. Intertemporal portfolio choice
7. Term structure & bond pricing

Utility theory: a brief summary

1. *Ordinal utility*
   - only meaningful to ask which option is better than the other
   - preference invariant to monotonic transformations
2. *Cardinal utility*
   - differences between preferences are also important.
   - preference invariant to positive affine transformations: $v(x) = au(x) + b$.
3. *Expected utility*
   - choice over lotteries
   - maximize the expectation of a *cardinal* utility function over states

Risk aversion

- Risk aversion = concavity of the utility function
- Measurements of risk aversion
  1. Relative risk aversion

$$R(W) = -\frac{u''(W)}{u'(W)} W$$

  2. Absolute risk aversion

$$A(W) = -\frac{u''(W)}{u'(W)}$$

- Power utility: $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$
  - Check: power utility is CRRA
  - Check: when $\gamma \to 1$, $u(W) = \lim_{\gamma \to 1} \frac{W^{1-\gamma}-1}{1-\gamma} = \ln(W)$ (l'Hôpital's rule)
- Exponential utility: $u(W) = -e^{-AW}$
  - Check: exponential utility is CARA

Risk premium and certainty equivalence

- Initial wealth W
- Risk premium π
  - amount willing to pay to avoid a zero mean risk $\tilde{x}$
  - indifferent between paying π and taking risk $\tilde{x}$

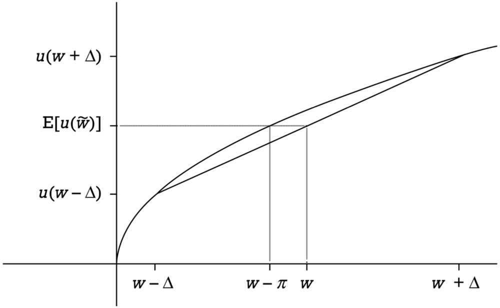
$$u(W-\pi)=Eu(W+\tilde{x})$$

- Certainty equivalence C^e
  - indifferent between accepting certain amount $W+C^e$ and taking risk $\mu+\tilde{x}$

$$u\left(W+C^e\right)=Eu(W+\mu+\tilde{x})$$

Risk premium and certainty equivalence

Example:



- Initial wealth w
- Zero mean risk $\tilde{x}$: $\frac{1}{2}$ chance to be $-\Delta$ and $\frac{1}{2}$ chance to be Δ
- Risk premium is π and certainty equivalence is $-\pi$.
- Question: what if the agent is risk seeking?

IMRS and EIS

- Start from Utility function $U(C_1, C_2)$ over two goods
- Marginal rate of substitution (MRS)

$$\mathrm{MRS}_{2,1} = \frac{\partial U/\partial C_2}{\partial U/\partial C_1} \quad \rightarrow \quad \mathrm{IMRS}_{t+1,t} = \frac{\partial U_t/\partial C_{t+1}}{\partial U_t/\partial C_t}$$

- **Finance is about trading goods among days**: $U(C_t, C_{t+1}, \cdots)$
- Intertemporal marginal rate of substitution (IMRS)
  - IMRS is a stochastic discount factor in consumption based asset pricing
- Elasticity of intertemporal substitution (EIS)

$$\mathrm{ES} = -\frac{\partial \log\left(\frac{C_2}{C_1}\right)}{\partial \log \mathrm{MRS}_{2,1}} \quad \rightarrow \quad \mathrm{EIS} = -\frac{\partial \log\left(\frac{C_{t+1}}{C_t}\right)}{\partial \log \mathrm{IMRS}_{t+1,t}}$$

Notations and Approximations

- Notations we will use during the whole semester
  - Level return of asset i is $R_{i,t+1}$
  - Log return of asset i is $r_{i,t+1} = \log(1 + R_{i,t+1})$
  - SDF M_{t+1}; Log SDF $m_{t+1} = \log(M_{t+1})$
  - Innovation (forcast error) $\tilde{x}_{t+1} = (E_{t+1} - E_t)x_{t+1}$
- Lognormality assumption: $r_{i,t+1} = \log(1 + R_{i,t+1}) \sim$ Normal
- Linear approximation

$$x \approx \log(x+1) \Leftrightarrow e^x \approx x + 1 \text{ when } x \approx 0$$

- Lognormal distribution: $x \sim N(\mu, \sigma^2) \Rightarrow e^x \sim \text{Lognormal}(\mu, \sigma^2)$

$$E_t e^x = \exp\left(E_t x + \frac{1}{2}\text{Var}_t(x)\right)$$

  - Check! (hint: use moment generating function.)

Notations and Approximations

- Assume lognormality: $r_{i,t+1} = \log(1 + R_{i,t+1}) \sim N(E_t r_{i,t+1}, \sigma^2_{it})$
- Risk premium

$$\begin{aligned} \text{RP} &= E_t R_{i,t+1} - R_{f,t+1} \\ &\approx E_t r_{i,t+1} + \frac{1}{2}\sigma^2_{it} - r_{f,t+1} \end{aligned}$$

- Derivation?

Notations and Approximations

- Assume lognormality: $r_{i,t+1} = \log(1 + R_{i,t+1}) \sim N(E_t r_{i,t+1}, \sigma_{it}^2)$
- Risk premium

$$
\begin{aligned}
\text{RP} &= E_t R_{i,t+1} - R_{f,t+1} \\
&\approx E_t r_{i,t+1} + \frac{1}{2}\sigma_{it}^2 - r_{f,t+1}
\end{aligned}
$$

- Derivation?

$$
\begin{aligned}
\text{RP} &= E_t R_{i,t+1} - R_{f,t+1} \\
&= E_t \left(R_{i,t+1} + 1\right) - 1 - R_{f,t+1} \\
&= E_t e^{\log\left(R_{i,t+1}+1\right)} - 1 - R_{f,t+1} \\
&= E_t e^{r_{i,t+1}} - 1 - R_{f,t+1} \\
&= e^{E_t r_{i,t+1} + \frac{1}{2}\sigma_{it}^2} - 1 - R_{f,t+1} \\
&\approx E_t r_{i,t+1} + \frac{1}{2}\sigma_{it}^2 + 1 - 1 - R_{f,t+1} \\
&\approx E_t r_{i,t+1} + \frac{1}{2}\sigma_{it}^2 - r_{f,t+1}
\end{aligned}
$$

Aside: lognormality of return in finance

Why do we typically work with (log) return in finance?

1. log return close to return: $r \equiv \log(1+R) \approx R$
2. fair to assume that price follows lognormal (no negative side) $\Rightarrow$ return is lognormal.
3. addictivity: $\log \frac{P_3}{P_1} = \log \frac{P_3}{P_2}\frac{P_2}{P_1} = \log \frac{P_2}{P_1} + \log \frac{P_3}{P_2} = r_1 + r_2$

Do returns really follow log-normal distribution?

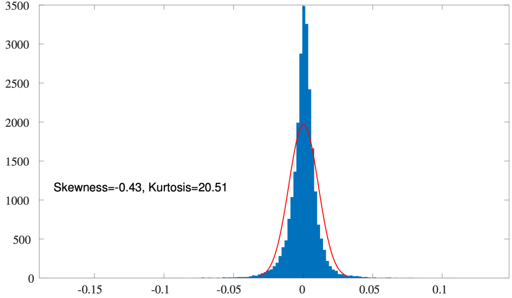


Figure: Distribution of historical daily market returns

Not a bad assumption, though peak and heavy tails.

Scenario 1: Principle of participation

Settings

- two periods: t, $t+1$, initial wealth W
- allocation θ to a risky asset with return $\tilde{R} = R_f + \tilde{x}$
- allocation $W - \theta$ to risk-free asset, with return R_f

Solution

- wealth at $t+1$:

$$\theta(1 + R_f + \tilde{x}) + (W - \theta)(1 + R_f) = W(1 + R_f) + \theta\tilde{x} \equiv W_0 + \theta\tilde{x}$$

- agent's problem: $V(\theta) = \max_\theta E_t u(W_0 + \theta\tilde{x})$
- FOC:

$$V'(\theta) = E_t \tilde{x} u'(W_0 + \theta\tilde{x}) = 0$$

Evaluate FOC at $\theta = 0$:

$$V'(0) = u'(W_0) E_t \tilde{x} > 0 \quad \text{if} \quad E_t \tilde{x} > 0$$

Principle of participation: one will participate in risky investment, at least a little bit, since utility function is local linear (local risk neutral).
But inconsistent to data, explanations: rational (fixed cost) and behavior explanations (prospect theory, pessimism and trust, financial literacy).

Scenario 2: Small risk case

Settings

- Same as in the last slide, except $\tilde{x} = k\mu + \tilde{y}$ and $E_t\tilde{y} = 0$

Solution

- Now FOC becomes

$$V'(\theta) = E_t\left[(k\mu + \tilde{y})u'\left(W_0 + \theta(k)(k\mu + \tilde{y})\right)\right] = 0$$

- Differentiate w.r.t k and evaluate at $k = 0$

$$\theta'(0) = \frac{\mu}{E_t\tilde{y}^2}\frac{1}{A\left(W_0\right)}, \quad \text{where} \quad A(W) = -\frac{u''(W)}{u'(W)}$$

- Approximate $\theta(k)$ surround 0 using Taylor expansion

$$\theta(k) \approx \theta(0) + k\theta'(0) = \frac{k\mu}{E_t\tilde{y}^2}\frac{1}{A\left(W_0\right)} = \frac{1}{A\left(W_0\right)}\frac{E_t\tilde{x}}{\operatorname{Var}_t\tilde{x}}$$

Scenario 3: CARA-normal case

Settings

- Almost the same as in the last slide
- CARA (exponential) utility $u(W) = -e^{-AW}$
- Normal excess return $\tilde{x} \sim N(\mu, \sigma^2)$

Solution

- agent's problem

$$V(\theta) = \max_{\theta} E_t u\left(W_0 + \theta \tilde{x}\right) = \min_{\theta} E_t e^{-A(W_0 + \theta \tilde{x})}$$

- since $-A\left(W_0 + \theta \tilde{x}\right)$ follows normal distribution, equivalent to

$$\min_{\theta} \exp\left(E_t\left[-A\left(W_0 + \theta \tilde{x}\right)\right] + \frac{1}{2}\mathrm{Var}_t\left[-A\left(W_0 + \theta \tilde{x}\right)\right]\right)$$

$$\Leftrightarrow \quad \min_{\theta} -A\theta\mu + \frac{1}{2}A^2\theta^2\sigma^2$$

- FOC gives

$$\theta = \frac{\mu}{A\sigma^2}$$

Scenario 4: CRRA-lognormal case

Settings

- CRRA utility $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$
- Lognormal return R_{t+1} and risk-free rate $R_{f,t+1}$
- Form a portfolio $(r_{p,t+1}, \sigma^2_{pt})$ by allocating the capital between
  1. one risky asset, share α_t, $r_{t+1} \sim N(E_t r_{t+1}, \sigma^2_t)$
  2. one risk-free asset, share $1-\alpha_t$, risk-free rate $r_{f,t+1}$
- Portfolio return

$$R_{p,t+1} = R_{f,t+1} + \alpha_t \left(R_{t+1} - R_{f,t+1}\right)$$

- Log portfolio return

$$r_{p,t+1} - r_{f,t+1} + \frac{\sigma^2_{pt}}{2} \approx \alpha_t \left(r_{t+1} - r_{f,t+1} + \frac{\sigma^2_t}{2} \right), \quad \text{with} \quad \sigma^2_{pt} = \alpha^2_t \sigma^2_t$$

  - Check it by your self after the class! Derivation in the appendix.
- Budget constraint

$$W_{t+1} = W_t \left(1 + R_{p,t+1}\right) \quad \Leftrightarrow \quad w_{t+1} = w_t + r_{p,t+1}$$

Scenario 4: CRRA-lognormal case

Solution

- agent solves

$$V(\alpha_t) = \max_{\alpha_t} E_t \frac{W_{t+1}^{1-\gamma}}{1-\gamma} = \max_{\alpha_t} E_t W_{t+1}^{1-\gamma} = \max_{\alpha_t} E_t \exp\left((1-\gamma)w_{t+1}\right)$$

- recall W_{t+1} follow log-normal distribution, equivalent to

$$\max_{\alpha_t} \exp\left((1-\gamma)E_t w_{t+1} + \frac{1}{2}(1-\gamma)^2 \sigma_{wt}^2\right)$$

$$\Leftrightarrow \max_{\alpha_t}(1-\gamma)E_t w_{t+1} + \frac{1}{2}(1-\gamma)^2 \sigma_{wt}^2$$

- Using budget constraint $w_{t+1} = w_t + r_{p,t+1}$

$$\max_{\alpha_t} E_t r_{p,t+1} + \frac{1}{2}(1-\gamma)\sigma_{pt}^2$$

- plugging into log portfolio return

$$\max_{\alpha_t} \alpha_t \mathrm{E}_t \left(r_{t+1} - r_{f,t+1} + \frac{\sigma_t^2}{2}\right) - \frac{\gamma}{2}\alpha_t^2 \sigma_t^2$$

- take FOC giving $\alpha_t \approx \frac{\mathrm{E}_t R_{t+1} - R_{f,t+1}}{\gamma \sigma_t^2}$.

Appendix

- CRRA utility $\rightarrow$ log utility when $\gamma \rightarrow 1$, (l'Hôpital's rule)

$$u(W) = \lim_{\gamma \to 1} \frac{W^{1-\gamma} - 1}{1-\gamma} = \lim_{\gamma \to 1} \frac{e^{(1-\gamma)\log W} - 1}{1-\gamma} = \lim_{\gamma \to 1} \log W e^{(1-\gamma)\log W} = \log W$$

- The moment generating function for normal distribution $x \sim N(\mu, \sigma^2)$

$$M(t) = Ee^{tx} = e^{t\mu + \frac{1}{2}\sigma^2 t^2}$$

Take $t = 1$ and get the mean for log normal distribution. Take $t = 2$ and get the variance.

Appendix

- Log portfolio return in CRRA-lognormal case:

$$
\begin{aligned}
r_{p,t+1} - r_{f,t+1} &= \log \frac{1 + R_{p,t+1}}{1 + R_{f,t+1}} = \log \frac{1 + R_{f,t+1} + \alpha_t (R_{t+1} - R_{f,t+1})}{1 + R_{f,t+1}} \\
&= \log \left[1 + \alpha_t \left(\frac{1 + R_{t+1}}{1 + R_{f,t+1}} - 1 \right) \right] \\
&= \log \left[1 + \alpha_t \left(\exp(r_{t+1} - r_{f,t+1}) - 1 \right) \right]
\end{aligned}
$$

- Take 2nd order Taylor expansion

$$
r_{p,t+1} - r_{f,t+1} \approx \alpha_t (r_{t+1} - r_{f,t+1}) + \frac{1}{2} \alpha_t (1 - \alpha_t)(r_{t+1} - r_{f,t+1})^2
$$

- Note $(r_{t+1} - r_{f,t+1})^2 \approx \sigma_t^2$ (see Campbell book P28 footnote), then

$$
r_{p,t+1} - r_{f,t+1} \approx \alpha_t (r_{t+1} - r_{f,t+1}) + \text{const} \quad \Leftrightarrow \quad \sigma_{p,t+1}^2 = \alpha_t^2 \sigma_{t+1}^2
$$