Asset Pricing Review Session 3

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Outlines for review sessions

First Part (Static Asset Pricing)

1. Choice under uncertainty
2. Static portfolio choice
3. Static asset pricing
4. Stochastic discount factor

Second Part (Intertemporal Asset Pricing)

1. Present value relations
2. Long run risk (BY)
3. Intertemporal CAPM (CV)
4. Rare Disaster (Martin)
5. Stochastic volatility (BKY and CGPT)
6. Intertemporal portfolio choice
7. Term structure & bond pricing

What is Stochastic Discount Factor?

- Finance is about time, risk, and information.
- SDF captures time → compares the goods between two periods.
- Suppose constant interest rate r, how to value payoff X of next period?

$$P = \underbrace{\frac{1}{1+r}}_{\text{Discount}} X$$

- However, we live in a world with randomness → r is not constant, depending on states
- We discounting differently in different state → the average discounting is stochastic discount factor
- Here we introduce SDF from consumption-based model

Portfolio Choice with Consumption: one Risky Asset

Settings

- Two periods, initial wealth W_t at t, non-financial income $\tilde{Y}_{t+1}$
- Utility function: $u\left(C_t\right)+\beta \mathrm{E}_t\left[u\left(\tilde{C}_{t+1}\right)\right]$
- One asset, price P_t, random payoff $\tilde{X}_{t+1}$, buy θ shares

Problem

$$\max_\theta u\left(C_t\right)+\beta \mathrm{E}_t\left[u\left(\tilde{C}_{t+1}\right)\right]$$

$$\text{s.t. } C_t=W_t-P_t\theta, \tilde{C}_{t+1}=\tilde{Y}_{t+1}+\tilde{X}_{t+1}\theta$$

Solution

$$P_t=\mathrm{E}_t\left[\beta\frac{u'\left(\tilde{C}_{t+1}\right)}{u'\left(C_t\right)}\tilde{X}_{t+1}\right] \qquad \widetilde{M}_{t+1}=\beta\frac{u'\left(\tilde{C}_{t+1}\right)}{u'\left(C_t\right)}=\frac{\partial U/\partial \tilde{C}_{t+1}}{\partial U/\partial C_t}$$

Portfolio Choice with Consumption: State Contingent Claims

Settings

- Two periods, initial wealth W_t at t
- S states in period $t+1$, non-financial income $Y_{t+1}(s)$ in state $s \in S$
- S state-contingent claims: asset s with price $q_t(s)$ and deliver 1 unit at state s in period $t+1$, portfolio choice $\{\theta(s)\}$,
- Utility function $U\left(C_t, C_{t+1}(s)\right)=u\left(C_t\right)+\beta \Sigma_{s \in S} \pi(s) u\left(C_{t+1}(s)\right)$

Problem

$$\begin{aligned}
\max_{\theta(s)} \quad & u\left(C_t\right)+\beta \Sigma_{s \in S} \pi_s u\left(C_{t+1}(s)\right) \\
\text{s.t. } & C_t=W_t-\Sigma_{s \in S} q_t(s) \theta(s) \\
& C_{t+1}(s)=Y_{t+1}(s)+\theta(s) \forall s \in S
\end{aligned}$$

Solution

$$q_t(s)=\pi(s) \beta \frac{u'\left(C_{t+1}(s)\right)}{u'\left(C_t\right)} \qquad M_{t+1}(s)=\beta \frac{u'\left(C_{t+1}(s)\right)}{u'\left(C_t\right)}=\frac{q_t(s)}{\pi(s)}$$

Pricing with SDF

Fundamental pricing equation

$$\begin{aligned} P_{i,t} &= \mathrm{E}_t\left[M_{t+1}X_{i,t+1}\right] = \mathrm{E}_t\left[M_{t+1}\left(D_{i,t+1}+P_{i,t+1}\right)\right] \\ 1 &= \mathrm{E}_t\left[M_{t+1}\left(1+R_{i,t+1}\right)\right] \end{aligned}$$

Riskfree rate

$$1 = \mathrm{E}_t\left[M_{t+1}\right]\left(1+R_{f,t+1}\right)$$

Risk Premium

$$\begin{aligned} & \mathrm{E}_t\left[M_{t+1}\right]\mathrm{E}_t\left[1+R_{i,t+1}\right] + \mathrm{Cov}_t\left(M_{t+1},R_{i,t+1}\right) = 1 \\ \Rightarrow\quad & \mathrm{E}_t\left[M_{t+1}\right]\mathrm{E}_t\left[R_{i,t+1}-R_{f,t+1}\right] + \mathrm{Cov}_t\left(M_{t+1},R_{i,t+1}-R_{f,t+1}\right) = 0 \\ \Rightarrow\quad & \mathrm{E}_t\left[M_{t+1}\right]\mathrm{E}_t\left[R_{i,t+1}-R_{f,t+1}\right] + \mathrm{Cov}_t\left(M_{t+1},R_{i,t+1}\right) = 0 \\ \Rightarrow\quad & \mathrm{E}_tR_{i,t+1}-R_{f,t+1} = -\frac{\mathrm{Cov}_t\left(M_{t+1},R_{i,t+1}\right)}{\mathrm{E}_tM_{t+1}} \\ & \qquad\qquad\qquad\quad = -\left(1+R_{f,t+1}\right)\mathrm{Cov}_t\left(M_{t+1},R_{i,t+1}\right) \end{aligned}$$

Pricing with SDF

Beta representation

- Risk premium

$$\mathrm{E}_t R_{i,t+1} - R_{f,t+1} = -\frac{\mathrm{Cov}_t\left(M_{t+1}, R_{i,t+1}\right)}{\mathrm{E}_t M_{t+1}}$$

- Pricing equation holds for all assets:

$$\frac{\mathrm{E}_t R_{i,t+1} - R_{f,t+1}}{\mathrm{Cov}_t\left(M_{t+1}, R_{i,t+1}\right)} = \frac{\mathrm{E}_t R_{j,t+1} - R_{f,t+1}}{\mathrm{Cov}_t\left(M_{t+1}, R_{j,t+1}\right)}$$

- Beta representation

$$\mathrm{E}_t R_{i,t+1} - R_{f,t+1} = \underbrace{\frac{\mathrm{Cov}_t\left(M_{t+1}, R_{i,t+1}\right)}{\mathrm{Var}_t\, M_{t+1}}}_{\equiv \beta_{iMt}} \underbrace{\left[-\frac{\mathrm{Var}_t\, M_{t+1}}{\mathrm{E}_t M_{t+1}}\right]}_{\equiv \lambda_{Mt}}$$

  1. β_{iMt}: quantity of risk, different for each asset
  2. λ_{Mt}: price of risk, same for all assets
- Intuition:
  - here: project $R_{i,t+1}$ into SDF.
  - CAPM: project $R_{i,t+1}$ into market portfolio.
  - APT: project $R_{i,t+1}$ into multiple factors

Definitions

Stochastic Discount Factor

Definition (Stochastic Discount Factor)

A stochastic discount factor is any random variable (process) M_t, such that, for any asset/portfolio i,

$$P_{i,t} = E_t[M_{t+1} X_{i,t+1}]$$

Law of One Price

Definition (Law of One Price)

If two assets/portfolios i, j have the same payoffs in every state, then they must have the same price.

$$X_i(s) = X_j(s) \forall s \implies P_i = P_j$$

$$P(aX_i + bX_j) = aP(X_i) + bP(X_j)$$

Definitions

Arbitrage Opportunity

Definition (Arbitrage Opportunity)

Under payoff space Ξ and pricing rule P, there is an arbitrage opportunity if $\exists X \in \Xi$ such that

1. $P(X) \leq 0$
2. $X \geq 0$ with probability 1
3. Either $P(X) < 0$ or $X(s) > 0$ for some state s (or both) with positive probability.

No Arbitrage

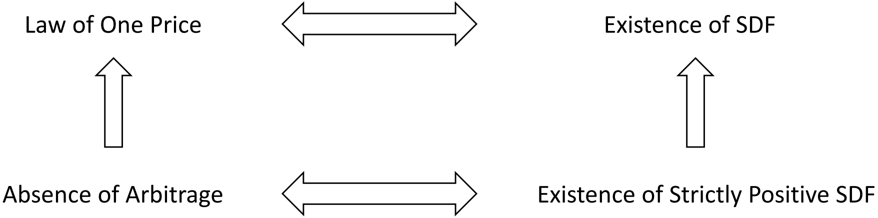
Definition (No Arbitrage)

Under payoff space Ξ and pricing rule P, there is no arbitrage opportunity if

1. $\forall X \in \Xi$ such that $X(s) \geq 0, \forall s$, we have $P(X) \geq 0$;
2. and $\forall X \in \Xi$ such that $X(s) \geq 0, \forall s$, and $X(s) > 0$ for some state s, we have $P(X) > 0$

Relation

Relation


Law of One Price

Existence of SDF

Absence of Arbitrage

Existence of Strictly Positive SDF

- John Campbell Section 4.2
- Kerry Back Section 3.2

SDF in Complete Market

Definition (Complete Market)

In a complete market, for each state s, there is a contingent claim (Arrow-Debreu security) that pays \$1 in state s and \$0 in any other state. Denote the price of such an asset as $q(s)$.

Assume law of one price $\Longleftrightarrow$ Existence of SDF

Find the SDF by construction

- Want to find a random variable M such that $P_i = E[MX_i], \forall i$
- Replicate the payoffs of i in each state s by a portfolio of AD securities
- The prices should be the same by the law of one price

$$P_i = \sum_s q(s) X_i(s) = \sum_s \pi(s) \frac{q(s)}{\pi(s)} X_i(s) = E\left[\underbrace{\frac{q(s)}{\pi(s)}}_{\equiv M(s)} X_i \right]$$

Aside: SDF in incomplete Market (1/2)

Incomplete Market

- S states
- N asset (no redundant assets); The market is incomplete since $N < S$
- Payoff $X = [X_1, \cdots, X_N]' \in R^N$; price $P = [P(X_1), \cdots, P(X_N)]' \in R^N$
- Space of tradable payoffs $\Xi = \left\{X'c : c \in \mathbb{R}^N\right\}$

Assume law of one price $\Longleftrightarrow$ Existence of SDF **Find the SDF by construction**

- Focus on the SDF X^* in the payoff space: $X^* = X'c^* \in \Xi$
- Plug into the SDF definition $P = E[XM] = E\left[XX'c^*\right]$
- Solve c^* and X^*

$$c^* = E\left[XX'\right]^{-1} P, \quad X^* = X'E\left[XX'\right]^{-1} P$$

Aside: SDF in incomplete Market (2/2)

- Properties of X^*
  - X^* always exists and is unique as long as $E[XX']$ is nonsingular
  - Can always derive an SDF using payoffs and prices!
  - X^* is tradable ($X^* \in \Xi$), a portfolio best mimics the behavior of every SDF
  - X^* is the SDF with minimum variance
- General SDF in incomplete market
  - Write any SDF as $M = X^* + \epsilon$ where $E[X\epsilon] = 0$
  - SDF = projection onto payoff space + residual orthogonal to payoff space
- Projection interpretation of X^*

$$\begin{aligned} X^* &= \operatorname{proj}(M \mid X) \\ &= X'\mathrm{E}\left[XX'\right]^{-1}\mathrm{E}[XM] \\ &= X'\mathrm{E}\left[XX'\right]^{-1} \end{aligned}$$

  - Linear projection model: $Y = X'\beta + \epsilon$ and $\mathrm{E}[X\epsilon] = 0$
  - $\beta = \mathrm{E}\left[XX'\right]^{-1}\mathrm{E}[XY], \operatorname{proj}(Y \mid X) = X'\beta = X'\mathrm{E}\left[XX'\right]^{-1}\mathrm{E}[XY]$

Volatility Bounds on SDF

The formula of risk premia

$$
\begin{aligned}
& \mathrm{E}R_i - R_f = -\frac{\operatorname{Cov}(M, R_i)}{\mathrm{E}M} \\
\Longrightarrow \quad & \mathrm{E}R_i - R_f = -\frac{\operatorname{Corr}(M, R_i)\,\sigma(M)\sigma(R_i)}{\mathrm{E}M} \\
\Longrightarrow \quad & \underbrace{\frac{\mathrm{E}R_i - R_f}{\sigma(R_i - R_f)}}_{\text{Sharpe Ratio}} = -\operatorname{Corr}(M, R_i)\,\frac{\sigma(M)}{\mathrm{E}M} \\
\Longrightarrow \quad & \left|\frac{\sigma(M)}{\mathrm{EM}}\right| \geq \underbrace{\left|\frac{\mathrm{E}R_i - R_f}{\sigma(R_i - R_f)}\right|}_{\text{Sharpe Ratio}}
\end{aligned}
$$

when does the equality holds?

- $|\operatorname{Corr}(M, R_i) = 1|$

Aside: Equality of Volatility Bounds

- Want to find an SDF M and an asset/portoflio i such that $|\operatorname{Corr}(M, R_i) = 1|$
- Recall the unique tradable SDF X^*
  - X^* is tradable with price P^* and payoff X^*

$$P^* = \mathrm{E}\left[MX^*\right] = \mathrm{E}\left[(X^*)^2\right], \quad 1 + R^* = \frac{X^*}{P^*} = \frac{X^*}{E\left[(X^*)^2\right]}$$

- Plug X^* and R^* into the correlation

$$\begin{aligned} |\operatorname{Corr}(X^*, R^*)| &= \left|\frac{\operatorname{Cov}(X^*, R^*)}{\sigma(X^*)\,\sigma(R^*)}\right| = \left|\frac{\operatorname{Cov}\left(X^*, \frac{X^*}{E\left[(X^*)^2\right]}\right)}{\sigma(X^*)\,\sigma\left(\frac{X^*}{E\left[(X^*)^2\right]}\right)}\right| \\ &= \left|\frac{\frac{1}{E\left[(X^*)^2\right]}\operatorname{Cov}(X^*, X^*)}{\frac{1}{E\left[(X^*)^2\right]}\sigma(X^*)\,\sigma(X^*)}\right| = 1 \end{aligned}$$

- X^* is the SDF with minimum variance
- R^* (the portfolio of X^*) has the highest Sharpe ratio.

Aside: Risk neutral measure

1. We start with some version of Euler equation $P_t = \mathbb{E}_t\left[M_{t+1}P_{t+1}\right]$, where M is the SDF.
    - Euler equation holds under very week assumptions (law of one price) and uses real-world probability.
    - We may find SDF by assuming some GE models, such as CAPM, C-CAPM, and investors' utility functions (CRRA, EZ utility)

2. The risk neutral measure Q is a hypothetical probability measure. Under this measure the discounted expected value of future price is exactly current prices.

$$P_t = \mathbb{E}_t\left[M_{t+1}P_{t+1}\right] = \frac{1}{1+r}\mathbb{E}_t^Q\left[P_{t+1}\right]$$

3. The intuition about Q: it's a merger of real-world probabilities with the risk preference. It corresponds to a world where all investors are risk-neutral and take Q prob. Comparing with real-world prob, risk-neutral probability is left-skwed (inflate the likelihood of bad events).

4. So the world is: people disagree on the real-world probability and risk-aversion level, but the market clears and there is one price. We can interpret the price as the result of risk-neutural investors and risk-neutral probability. And we can use risk-neutral probability to price derivatives.

Aside: Risk neutral measure in practice

We start with a one period model to derive risk-neutral probability measure, then we price a derivative use the Q measure.

1. Derive risk-neutral measure in a simple model.
    - Assume today's stock price is S_0, and stock price tomorrow can be uS_0 or dS_0. Assume gross risk-free rate r.
    - Based on non arbitrage theory, assume risk-neutral measure (p_u, p_d), the it should be

$$S_0 = \frac{1}{r}(uS_0p_u + dS_0p_d)$$

    - As $p_u + p_d = 1$, we have risk-neutral measure

$$p_u = \frac{r-d}{u-d}, p_d = \frac{u-r}{u-d}$$

and

$$S_0 = \frac{1}{r}E^Q[S_1]$$

Aside: Risk neutral measure in practice

We start with a one period model to derive risk-neutral probability measure, then we price a derivative use the Q measure.

2. Price a derivative.
   - Here is a derivative on the stock with payoff function $V(S_1)$. We can replicate the derivative payoff in both state with a profolio of risk free bond and stock:

$$\begin{aligned} xS_u + yr &= V\left(S_u\right) \\ xS_d + yr &= V\left(S_d\right) \end{aligned}$$

   - It gives profolio weights

$$x = \frac{V\left(S_u\right) - V\left(S_d\right)}{S_0(u-d)}, \quad y = \frac{uV\left(S_d\right) - dV\left(S_u\right)}{(u-d)}\frac{1}{r}$$

   - Based on law of one price, the derivative price at t_0 should be

$$V(S_0, t_0) = xS_0 + y \times 1 = \frac{1}{r}\left(V\left(S_u\right)\left(\frac{r-d}{u-d}\right) + V\left(S_d\right)\left(\frac{u-r}{u-d}\right)\right) = \frac{1}{r}E^Q[V(S_1)]$$

   - Ha, instead of calculate the replication portfolio weights, we can use risk-neutral measure to price the derivatives!
3. We can extend to continuous-time models, and get BS model. And from observed market data, we can estimate risk-neutral probability (twice differentiating the implied volatility surface with respect to strike).

Case 1: quadratic utility function

Settings

- Two periods, initial wealth W_t at t
- N assets, w_i, $R_{i,t+1}$, form wealth portfolio $R_{W,t+1} = \sum_i w_i R_{i,t+1}$
- Utility function $U\left(C_t, C_{t+1}\right) = -\frac{1}{2}\left(\bar{C} - C_t\right)^2 - \frac{1}{2}\beta \mathrm{E}_t\left(\bar{C} - C_{t+1}\right)^2$

Problem

$$
\begin{aligned}
U\left(C_t, C_{t+1}\right) &= -\frac{1}{2}\left(\bar{C} - C_t\right)^2 - \frac{1}{2}\beta \mathrm{E}_t\left(\bar{C} - C_{t+1}\right)^2 \\
\text{s.t. } C_{t+1} &= W_{t+1}, \quad W_{t+1} = \left(1 + R_{W,t+1}\right)\left(W_t - C_t\right)
\end{aligned}
$$

SDF

$$
M_{t+1} = \frac{\partial U / \partial C_{t+1}}{\partial U / \partial C_t} = \beta \frac{\bar{C} - C_{t+1}}{\bar{C} - C_t} = \underbrace{\frac{\beta \bar{C}}{\bar{C} - C_t} - \frac{\beta\left(1 + R_{W,t+1}\right)\left(W_t - C_t\right)}{\bar{C} - C_t}}_{\equiv a_t - b_t R_{W,t+1}}
$$

Case 1: quadratic utility function

Plug SDF into the pricing equation of risk premium

$$\frac{\mathrm{E}_t R_{i,t+1} - R_{f,t+1}}{\mathrm{Cov}_t\left(M_{t+1}, R_{i,t+1}\right)} = \frac{\mathrm{E}_t R_{W,t+1} - R_{f,t+1}}{\mathrm{Cov}_t\left(M_{t+1}, R_{W,t+1}\right)}$$

$$\iff \quad \mathrm{E}_t R_{i,t+1} - R_{f,t+1} = \frac{\mathrm{Cov}_t\left(M_{t+1}, R_{i,t+1}\right)}{\mathrm{Cov}_t\left(M_{t+1}, R_{W,t+1}\right)} \left[\mathrm{E}_t R_{W,t+1} - R_{f,t+1}\right]$$

Arrange the equation, get

$$\begin{aligned}
\mathrm{E}_t R_{i,t+1} - R_{f,t+1} &= \frac{-b_t \,\mathrm{Cov}_t\left(R_{W,t+1}, R_{i,t+1}\right)}{-b_t \,\mathrm{Cov}_t\left(R_{W,t+1}, R_{W,t+1}\right)} \left[\mathrm{E}_t R_{W,t+1} - R_{f,t+1}\right] \\
&= \frac{\mathrm{Cov}_t\left(R_{W,t+1}, R_{i,t+1}\right)}{\mathrm{Var}_t\left(R_{W,t+1}\right)} \left[\mathrm{E}_t R_{W,t+1} - R_{f,t+1}\right] \\
&= \beta_{iW\,t} \left[\mathrm{E}_t R_{W,t+1} - R_{f,t+1}\right]
\end{aligned}$$

Case 2: CRRA utility function -Lognormal return

Settings

- Two periods, initial wealth W_t at t
- N assets, $\{w_i\}$, $R_{i,t+1}$, form wealth portfolio $R_{W,t+1} = \sum_i w_i R_{i,t+1}$
- Utility function $U(C_t, C_{t+1}) = \frac{C_t^{1-\gamma}-1}{1-\gamma} + \beta \mathrm{E}_t \frac{C_{t+1}^{1-\gamma}-1}{1-\gamma}$

Problem

$$U(C_t, C_{t+1}) = \frac{C_t^{1-\gamma}-1}{1-\gamma} + \beta \mathrm{E}_t \frac{C_{t+1}^{1-\gamma}-1}{1-\gamma}$$

$$\text{s.t. } C_{t+1} = W_{t+1}, \quad W_{t+1} = (1 + R_{W,t+1})(W_t - C_t)$$

SDF

$$M_{t+1} = \frac{\partial U / \partial C_{t+1}}{\partial U / \partial C_t} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} = \beta \left(\frac{(1 + R_{W,t+1})(W_t - C_t)}{C_t} \right)^{-\gamma}$$

$$m_{t+1} = \log \beta - \gamma \Delta c_{t+1} = \underbrace{\log \beta - \gamma \left(r_{W,t+1} + \log \frac{W_t - C_t}{C_t} \right)}_{\equiv -\gamma r_{W,t+1} + \text{const}}$$

Case 2: CRRA utility function -Lognormal return

Plug log SDF into pricing equation

- Start with log pricing equation

$$\mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma_{it}^2 - r_{f,t+1} = -\operatorname{Cov}_t\left(r_{i,t+1}, m_{t+1}\right)$$

- Plug into log SDF

$$\mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma_{it}^2 - r_{f,t+1} = \gamma \operatorname{Cov}_t\left(r_{i,t+1}, r_{W,t+1}\right)$$

- Above pricing equation also holds for wealth portfolio

$$\mathrm{E}_t r_{W,t+1} + \frac{1}{2}\sigma_{Wt}^2 - r_{f,t+1} = \gamma \operatorname{Cov}_t\left(r_{W,t+1}, r_{W,t+1}\right)$$

- Connect above two equations to get beta representation:

$$\begin{aligned}\mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma_{it}^2 - r_{f,t+1} &= \frac{\operatorname{Cov}_t\left(r_{i,t+1}, r_{W,t+1}\right)}{\operatorname{Var}_t\left(r_{W,t+1}\right)}\left[\mathrm{E}_t r_{W,t+1} + \frac{1}{2}\sigma_{Wt}^2 - r_{f,t+1}\right] \\ &= \beta_{iWt}\left[\mathrm{E}_t r_{W,t+1} + \frac{1}{2}\sigma_{Wt}^2 - r_{f,t+1}\right]\end{aligned}$$

General Form

General form of a K-factor model

$$(\forall i)\mathrm{E}R_i - R_f = \lambda' \beta_i = \Sigma_k \lambda_k \beta_{k,i}$$

- K-factor vector $f = [f_1 \dots f_K]'$, with covariance matrix Σ_f
- $\beta_i = [\beta_{1,i} \dots \beta_{K,i}]'$: quantity of risk

$$\beta_i = \mathrm{Var}(f)^{-1} \, \mathrm{Cov}\,(f, R_i) = \Sigma_f^{-1} \, \mathrm{Cov}\,(f, R_i)$$

- $\lambda = [\lambda_1 \dots \lambda_K]'$: price of risk
- Expected excess return = Quantity of risk × Price of risk

Factor Risk Premium

Why is λ called the factor risk premium?

- Assume factor f_j is tradable (f_j is the return of an asset)

$$\begin{aligned} & \mathrm{E} f_j - R_f = \lambda' \beta_j \\ \Longrightarrow \quad & \mathrm{E} f_j - R_f = \lambda' \Sigma_f^{-1} \operatorname{Cov}(f, f_j) \\ \Longrightarrow \quad & \mathrm{E} f_j - R_f = \lambda' e_j = \lambda_j \\ \Longrightarrow \quad & \beta_i = \Sigma_f^{-1} \operatorname{Cov}(f, R_i) \end{aligned}$$

- e_j is the j selection vector
- λ_j is the excess return of asset/factor f_j (factor risk premium)

SDF of Factor Models

Suppose the world follows Factor models, what is the SDF that makes such world?

- Guess and verify (equating coefficients): $M = a + b'(f - Ef)$
- Check the first moment (risk-free rate)

$$\mathrm{E}M = \mathrm{E}a + b'\mathrm{E}(f - \mathrm{E}f) = a = \frac{1}{R_f} \quad \Longrightarrow \quad a = \frac{1}{R_f}$$

- Check the second moment (risk premia)

$$\begin{aligned}\mathrm{E}R_i - R_f &= -\frac{\operatorname{Cov}(M, R_i)}{\mathrm{E}M} \\ &= -\frac{\operatorname{Cov}(a + b'(f - \mathrm{E}f), R_i)}{a} = -\frac{1}{a}b'\operatorname{Cov}(f, R_i) \\ &= -\frac{1}{a}b'\Sigma_f\Sigma_f^{-1}\operatorname{Cov}(f, R_i) = -\frac{1}{a}b'\Sigma_f\beta_i\end{aligned}$$

Note the formula of factor model: $\mathrm{E}R_i - R_f = \lambda'\beta_i$, hence

$$\lambda' = -\frac{1}{a}b'\Sigma_f, \quad b = -\frac{1}{R_f}\Sigma_f^{-1}\lambda \quad \Longrightarrow \quad M = \frac{1}{R_f} - \frac{1}{R_f}\lambda'\Sigma_f^{-1}(f - \mathrm{E}f)$$

Aside 1

$$
\begin{aligned}
&\Sigma_f^{-1}\operatorname{Cov}(f, f_j) \\
&= \begin{bmatrix} \operatorname{Var}(f_1) & \cdots & \operatorname{Cov}(f_1, f_j) & \cdots & \operatorname{Cov}(f_1, f_K) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \cdots & \operatorname{Var}(f_j) & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \operatorname{Cov}(f_K, f_1) & \cdots & \operatorname{Cov}(f_K, f_j) & \cdots & \operatorname{Var}(f_K) \end{bmatrix}^{-1} \times \begin{bmatrix} \operatorname{Cov}(f_1, f_j) \\ \vdots \\ \operatorname{Var}(f_j) \\ \vdots \\ \operatorname{Cov}(f_K, f_j) \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = e_j
\end{aligned}
$$

Aside 2

$$\begin{aligned}\operatorname{Var}(R_m) &= \operatorname{Var}\left(\sum_{i=1}^{n} w_i R_i\right) = \operatorname{Cov}\left(\sum_{i=1}^{n} w_i R_i, \sum_{i=1}^{n} w_i R_i\right)\\ &= \sum_{i=1}^{n}\sum_{j\neq i}^{n} w_i w_j \operatorname{Cov}\left(R_i, R_j\right) + \sum_{i}^{n} w_i^2 \operatorname{Var}(R_i)\end{aligned}$$

$$\begin{aligned}\frac{\partial \operatorname{Var}(R_m)}{\partial w_i} &= 2\sum_{j\neq i}^{n} w_j \operatorname{Cov}\left(R_i, R_j\right) + 2w_i \operatorname{Cov}\left(R_i, R_i\right)\\ &= 2\operatorname{Cov}\left(R_i, R_m\right)\end{aligned}$$