Asset Pricing Review Session 4

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Outlines for review sessions

First Part (Static Asset Pricing)

1. Choice under uncertainty
2. Static portfolio choice
3. Static asset pricing
4. Stochastic discount factor

Second Part (Intertemporal Asset Pricing)

1. Present value relations
2. Long run risk (BY)
3. Intertemporal CAPM (CV)
4. Rare Disaster (Martin)
5. Stochastic volatility (BKY and CGPT)
6. Intertemporal portfolio choice
7. Term structure & bond pricing

Roadmap

1. Refinements on logs and approximations
   - Risk premium, pricing equation, portfolio choice
   - Campbell Shiller approximation and decompositions
2. CCAPM and Three asset pricing puzzles
3. Epstein-Zin utility
   - Review power utility
   - Derivation of Epstein-Zin Utility and intuitions
   - Properties of Epstein-Zin Utility
   - SDF of Epstein-Zin Utility (using wealth portfolio)
   - Pricing riskfree rate and risk premium using SDF
4. Epstein-Zin with return decompositions
   - Extended consumption CAPM (long run risk, BY)
   - Intertemporal CAPM (good beta and bad beta, CV)

Approximations (1/6)

Notations

- Raw return of asset i is $R_{i,t+1}$
- Log return of asset i is $r_{i,t+1} = \log(1 + R_{i,t+1})$
- Log SDF $m_{t+1} = \log(M_{t+1})$
- Innovation $\tilde{x}_{t+1} = (\mathrm{E}_{t+1} - \mathrm{E}_t)\, x_{t+1}$

Approximation

$$x \approx \log(x+1) \text{ when } x \approx 0$$

Expectation of lognormal variables

$$\mathrm{E}_t e^x = e^{\mathrm{E}_t x + \frac{1}{2} \mathrm{Var}_t\, x}$$

Approximations (2/6)

Risk premium

$$\begin{aligned}\mathrm{RP} &= \mathrm{E}_t R_{i,t+1} - R_{f,t+1} \\ &\approx \mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma_{it}^2 - r_{f,t+1}\end{aligned}$$

Pricing Equation

$$\mathrm{E}_t\left[M_{t+1}\left(1+R_{i,t+1}\right)\right] = 1$$

$$\xrightarrow{\text{Taking logs}} \quad \mathrm{E}_t m_{t+1} + \frac{1}{2}\sigma_{mt}^2 + \mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma_{it}^2 + \sigma_{imt} = 0$$

Under joint lognormality of SDF and returns (need to show!)

Approximations (3/6)

Price the riskfree asset

$$\begin{aligned} \mathrm{E}_t m_{t+1} + \frac{1}{2}\sigma^2_{mt} + \mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma^2_{it} + \sigma_{imt} &= 0 \\ \mathrm{E}_t m_{t+1} + \frac{1}{2}\sigma^2_{mt} + r_{f,t+1} + 0 + 0 &= 0 \\ r_{f,t+1} &= -\mathrm{E}_t m_{t+1} - \frac{1}{2}\sigma^2_{mt} \end{aligned}$$

Price risky asset i

$$\begin{aligned} \mathrm{E}_t m_{t+1} + \frac{1}{2}\sigma^2_{mt} + \mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma^2_{it} + \sigma_{imt} &= 0 \\ -r_{f,t+1} + \mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma^2_{it} + \sigma_{imt} &= 0 \\ \mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma^2_{it} - r_{f,t+1} &= -\sigma_{imt} \end{aligned}$$

Approximations (4/6)

Portfolio choice with log returns

$$r_{p,t+1} - r_{f,t+1} + \frac{\sigma_{pt}^2}{2} \approx \alpha_t \left(r_{t+1} - r_{f,t+1} + \frac{\sigma_t^2}{2} \right)$$

$$\sigma_{pt}^2 = \alpha_t^2 \sigma_t^2 \quad \text{Cov}_t \left(r_{p,t+1}, r_{t+1} \right) = \alpha_t \sigma_t^2$$

- Under lognormality
- Form a portfolio $\left(r_{p,t}, \sigma_{pt}^2 \right)$ by allocating the capital between
  - One riskfree asset, $1 - \alpha_t, r_{f,t}$
  - One risky asset, $\alpha_t, r_t, \sigma_t^2$

Approximations (5/6)

Campbell Shiller approximation

$$r_{t+1} = \rho p_{t+1} + (1-\rho) d_{t+1} - p_t + k$$

- "Weighted average" of (log) future payoff

Price decomposition

$$p_t = \frac{k}{1-\rho} + \sum_{j=0}^{\infty} \rho^j (1-\rho) d_{t+1+j} - \sum_{j=0}^{\infty} \rho^j r_{t+1+j}$$

- Foundation of other decompositions
- Imposing the no bubble condition $\lim_{j\to\infty} \rho^j p_{t+j} = 0$

Approximations (6/6)

Dividend price ratio decomposition

$$d_t - p_t = \frac{-k}{1-\rho} + \sum_{j=0}^{\infty} \rho^j \left(-\Delta d_{t+1+j}\right) + \sum_{j=0}^{\infty} \rho^j r_{t+1+j}$$

Return decomposition

$$\tilde{r}_{t+1} = \underbrace{(\mathrm{E}_{t+1} - \mathrm{E}_t) \sum_{j=0}^{\infty} \rho^j \Delta d_{t+1+j}}_{\equiv N_{CF,t+1}} - \underbrace{(\mathrm{E}_{t+1} - \mathrm{E}_t) \sum_{\color{red}{j=1}}^{\infty} \rho^j r_{t+1+j}}_{\equiv N_{DR_{t+1}}}$$

- $N_{CF,t+1}$: news about cash flows
- $N_{DR_{t+1}}$: news about discount rates
- Can be applied to any asset/portfolio (e.g., the wealth portfolio in E-Z)

Power Utility

Definition

$$u\left(C_t\right) = \frac{C_t^{1-\gamma} - 1}{1-\gamma}$$

Properties

- CRRA: constant relative risk aversion
- EIS is $1/\gamma$; For $\gamma = 1$, log utility
- Time separable

SDF

$$M_{t+1} = \delta \frac{u'\left(C_{t+1}\right)}{u'\left(C_t\right)} \quad m_{t+1} = \log \delta - \gamma \Delta c_{t+1}$$

CCAPM (1/2)

Assumptions: Δc_{t+1} and r_{t+1} are

1. jointly conditionally lognormal
2. homoskedastic: All second moments are constant, aka dropping t or all (co)variance terms.

Riskfree rate

$$r_{f,t+1} = -\mathrm{E}_t m_{t+1} - \frac{1}{2}\sigma^2_{mt} \Longrightarrow r_{f,t+1} = -\log\delta + \gamma \mathrm{E}_t \Delta c_{t+1} - \frac{1}{2}\gamma^2 \sigma_c^2$$

1. First term: compensation for time
   - $\delta \in (0,1)$, low $\delta \rightarrow$ consumption tomorrow discounts more
2. Second term: consumption smooth incentive
   - high $\gamma \rightarrow$ low EIS $\rightarrow$ high consumption smooth incentive $\rightarrow$ borrow more
3. Third term: precautionary saving incentive
   - high $\gamma \rightarrow$ high risk aversion $\rightarrow$ high precautionary saving incentive $\rightarrow$ save more

CCAPM (2/2)

Risk premium

$$\mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma_{it}^2 - r_{f,t+1} = -\sigma_{imt} \Longrightarrow \mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma_i^2 - r_{f,t+1} = \gamma\sigma_{ic}$$

Risk premium only determined by

- price of risk, measured by risk tolerance γ
- quantity of risk σ_{ic} (bonds, stock, and insurance)

Three puzzles

- Equity premium puzzle: need too high γ to generate real premium
- Risk-free rate puzzle: if we adopt high γ, risk-free rate too high
- Volatility puzzle: P/D fluctuates too much.

Solutions

1. Rational, frictionless model
    - Habit formation (Campbell and Cochrane, 1999)
    - Long-run risk model (Bansal and Yaron, 2004; Campbell ICCAPM “good beta, bad beta”; Bansal, Kiku, Yaron, 2012)
    - Rate diasters model (Barro, 2006; Gabaix, 2012; Martain, 2013; Wachter, 2013)
    - Stochastic Volatility model (CGPT model)
    - also: Learning (Timmermann, 1993)
2. Behavior explanations....

Epstein-Zin Utility (1/10)

Want to separate risk aversion vs EIS forces?

1. Start with

$$U_t = f\left(C_t, \mathcal{CE}_t\left(U_{t+1}\right)\right)$$

- where $\mathcal{CE}(\cdot)$ is a certainty equivalence function $\mathcal{CE}_t(x) = G^{-1}\left(\mathrm{E}_t G(x)\right)$
- Let $G(\cdot)$ have a power form $G(x) = x^{1-\gamma}$, then $\mathcal{CE}_t(x) = \left(\mathrm{E}_t x^{1-\gamma}\right)^{\frac{1}{1-\gamma}}$

2. Then

$$U_t = f\left(C_t, \left(\mathrm{E}_t U_{t+1}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}\right)$$

- Let $f(\cdot)$ be CES (constant EIS)

$$f\left(x_1, x_2\right) = \left\{(1-\delta)\left(x_1\right)^{\rho} + \delta\left(x_2\right)^{\rho}\right\}^{\frac{1}{\rho}} \qquad \psi = \epsilon = \frac{1}{1-\rho} \Rightarrow \rho = 1 - \frac{1}{\psi}$$

3. Finally

$$U_t = \left\{(1-\delta)\left(C_t\right)^{1-\frac{1}{\psi}} + \delta\left(\mathrm{E}_t\left(U_{t+1}^{1-\gamma}\right)\right)^{\frac{1-1/\psi}{1-\gamma}}\right\}^{\frac{1}{1-\frac{1}{\psi}}}$$

Epstein-Zin Utility (2/10)

Definition

$$U_t = \left\{ (1-\delta)\left(C_t\right)^{1-\frac{1}{\psi}} + \delta\left(\mathrm{E}_t\left(U_{t+1}^{1-\gamma}\right)\right)^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\psi}}}$$

$$U_t = \left\{ (1-\delta)\left(C_t\right)^{\frac{1-\gamma}{\theta}} + \delta\left(\mathrm{E}_t\left(U_{t+1}^{1-\gamma}\right)\right)^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}}, \quad \theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$$

Properties

1. Time non-separable
2. EIS is ψ; RRA is γ
   - For $\gamma = 1/\psi$, power utiltiy;
   - For $\gamma = 1/\psi = 1$, log utility.
3. $\psi = 1$, constant consumption wealth ratio;
   $\gamma = 1$, myopic portfolio choice
4. If $\gamma > 1/\psi$, prefer early resolution of uncertainty;
   If $\gamma < 1/\psi$, prefer late resolution of uncertainty Campbell textbook section 6.4

Epstein-Zin Utility (3/10)

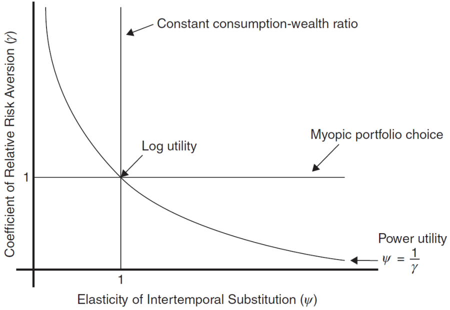

Figure 6.2. *Epstein-Zin Parameter Space*

- $\psi = 1$, constant consumption wealth ratio;
  $\gamma = 1$, myopic portfolio choice: investment horizon has no effects on portfolio choice.
- If $\gamma > 1/\psi$, prefer early resolution of uncertainty (BY);
  If $\gamma < 1/\psi$, prefer late resolution of uncertainty
  - Early resolution of uncertainty: transform uncertainty into predictable variation in future consumption

Epstein-Zin Utility (4/10)

Constant elasticity of substitution (CES)

- In multi period case, constant elasticity of intertemporal substitution ψ
- Use CES to integrate today's consumption and CE of future's consumption

Extreme Cases

- If $\psi = 1$ (unit EIS, $\rho = 0$), CES is Cobb-Douglas

$$U_t = (C_t)^{(1-\delta)} \left(\mathrm{E}_t \left(U_{t+1}^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \right)^{\delta}$$

- If $\psi = 0$ ($\rho = -\infty$), CES is Leontief

$$U_t = \min \left\{ (1-\delta) C_t, \delta \mathrm{E}_t \left(U_{t+1}^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \right\}$$

- If $\psi = \infty$ ($\rho = 1$), CES is linear

$$U_t = (1-\delta) C_t + \delta \mathrm{E}_t \left(U_{t+1}^{1-\gamma} \right)^{\frac{1}{1-\gamma}}$$

Epstein-Zin Utility (5/10)

Derive the SDF

1. Denote $\mathcal{CE}_t(U_{t+1}) \equiv E_t[U_{t+1}^{1-\gamma}]^{1/1-\gamma}$. Then the EZ aggregator:

$$U_t = \left\{(1-\delta)\left(C_t\right)^{1-\frac{1}{\psi}} + \delta\left[\mathcal{CE}_t(U_{t+1})\right]^{1-\frac{1}{\psi}}\right\}^{\frac{1}{1-\frac{1}{\psi}}}$$

2. From the EZ aggregator, calculate $MC_t \& MU_t$

$$\frac{\partial U_t}{\partial C_t} = (1-\delta)U_t^{1/\psi}C_t^{-1/\psi}$$

$$\frac{\partial U_t}{\partial U_{t+1}} = \delta U_t^{1/\psi}\left[\mathcal{CE}_t\left(U_{t+1}\right)\right]^{-1/\psi} U_{t+1}^{-\gamma}\left[\mathcal{CE}_t\left(U_{t+1}\right)\right]^{\gamma}$$

3. Plugging $MC_t \& MU_t$ into SDF

$$M_{t+1} = \frac{\frac{\partial U_t}{\partial C_{t+1}}}{\frac{\partial U_t}{\partial C_t}} = \frac{\frac{\partial U_t}{\partial U_{t+1}}\frac{\partial U_{t+1}}{\partial C_{t+1}}}{\frac{\partial U_t}{\partial C_t}} = \underbrace{\delta\left(\frac{C_{t+1}}{C_t}\right)^{-1/\psi}}_{\text{consumption growth term}} \underbrace{\left[\frac{U_{t+1}}{\mathcal{CE}_t\left(U_{t+1}\right)}\right]^{\frac{1}{\psi}-\gamma}}_{\text{innovations in continuation utility}}$$

- $1/\psi = \gamma \rightarrow$ standard time-additive SDF.
- Next step: use wealth portfolio (market portfolio) to build proxies for innovations in continuation utility $U_{t+1}/\mathcal{CE}_t\left(U_{t+1}\right)$

Epstein-Zin Utility (6/10)

4. Build proxies for $U_{t+1}/\mathcal{CE}_t\left(U_{t+1}\right)$ with wealth portfolio

a. Wealth portfolio return denotes $R_{w,t+1}$, subject to budget constraint

$$W_{t+1} = \left(1 + R_{w,t+1}\right)\left(W_t - C_t\right) \tag{1}$$

Pricing equation

$$\mathrm{E}_t\left[M_{t+1}\left(1 + R_{w,t+1}\right)\right] = 1 \iff W_t = C_t + \mathrm{E}_t\left[M_{t+1}W_{t+1}\right]$$

b. Note EZ utility has constant return to scale $U_t\left(\lambda C_t, \lambda U_{t+1}\right) = \lambda U_t\left(C_t, U_{t+1}\right) \Rightarrow$ Euler Theorem applies

$$U_t\left(C_t, U_{t+1}\right) = \frac{\partial U_t}{\partial C_t}C_t + \mathrm{E}_t\left(\frac{\partial U_t}{\partial U_{t+1}}U_{t+1}\right)$$

c. And recall that **SDF is intertemporal marginal rate of substitution (IMRS)**

$$M_{t+1} = \mathrm{IMRS}_{t+1,t} = \frac{\partial U_t/\partial C_{t+1}}{\partial U_t/\partial C_t} = \frac{\left(\partial U_t/\partial U_{t+1}\right)\left(\partial U_{t+1}/\partial C_{t+1}\right)}{\partial U_t/\partial C_t}$$

Epstein-Zin Utility (6/10)

4. Build proxies for $U_{t+1}/\mathcal{CE}_t\left(U_{t+1}\right)$ with wealth portfolio
   d. $\frac{U_t}{\left(\partial U_t/\partial C_t\right)}$ is exactly agent's wealth W_t

$$U_t\left(C_t,U_{t+1}\right)=\frac{\partial U_t}{\partial C_t}C_t+\mathrm{E}_t\left(\frac{\partial U_t}{\partial U_{t+1}}U_{t+1}\right)$$

$$\xrightarrow[\partial U_t/\partial C_t]{\text{divided by}}\frac{U_t}{\left(\partial U_t/\partial C_t\right)}=C_t+\mathrm{E}_t\left[\frac{\left(\partial U_t/\partial U_{t+1}\right)}{\left(\partial U_t/\partial C_t\right)}U_{t+1}\right]$$

$$\xrightarrow[M_{t+1}]{\text{Construct}}\frac{U_t}{\left(\partial U_t/\partial C_t\right)}=C_t+\mathrm{E}_t\left[\underbrace{\frac{\left(\partial U_t/\partial U_{t+1}\right)\left(\partial U_{t+1}/\partial C_{t+1}\right)}{\left(\partial U_t/\partial C_t\right)}}_{\equiv M_{t+1}}\frac{U_{t+1}}{\left(\partial U_{t+1}/\partial C_{t+1}\right)}\right]$$

$$\xrightarrow[W_t=C_t+\mathrm{E}_t\left(M_{t+1}W_{t+1}\right)]{\text{Compare with}}W_t=\frac{U_t}{\left(\partial U_t/\partial C_t\right)}$$

Then consumption to wealth ratio

$$\frac{W_t}{C_t}=\frac{1}{1-\delta}\left(\frac{U_t}{C_t}\right)^{1-1/\psi} \tag{2}$$

- It comes from MC$_t$: $\frac{\partial U_t}{\partial C_t}=\left(1-\delta\right)\left(\frac{U_t}{C_t}\right)^{1/\psi}$
- Constant Constant consumption to wealth ratio if $\psi=1$

Epstein-Zin Utility (6/10)

4. Build proxies for $U_{t+1}/\mathcal{CE}_t\left(U_{t+1}\right)$ with wealth portfolio

 d. Plug EZ formular $U_t=\left\{(1-\delta)\left(C_t\right)^{1-\frac{1}{\psi}}+\delta\left[\mathcal{CE}_t(U_{t+1})\right]^{1-\frac{1}{\psi}}\right\}^{\frac{1}{1-\frac{1}{\psi}}}$ into (2)

$$W_t-C_t=\frac{C_t}{1-\delta}\delta\left[\frac{\mathcal{CE}_t(U_{t+1})}{C_t}\right]^{1-\frac{1}{\psi}} \tag{3}$$

 Recall return on wealth portfolio is given by

$$1+R_{w,t+1}=\frac{W_{t+1}}{W_t-C_t}$$

 Replace W_{t+1} with (2) and W_t-C_t with (3). Then, $U_{t+1}/\mathcal{CE}_t\left(U_{t+1}\right)$ can be approximated by wealth portfolio return

$$1+R_{w,t+1}=\frac{1}{\delta}\left(\frac{C_{t+1}}{C_t}\right)^{\frac{1}{\psi}}\left[\frac{U_{t+1}}{\mathcal{CE}_t\left(U_{t+1}\right)}\right]^{1-\frac{1}{\psi}}$$

$$\Rightarrow\quad \frac{U_{t+1}}{\mathcal{CE}_t\left(U_{t+1}\right)}=\delta^{\frac{\psi}{\psi-1}}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi-1}}\left(1+R_{w,t+1}\right)^{\frac{\psi}{\psi-1}}$$

5. Finally, plug $U_{t+1}/\mathcal{CE}_t\left(U_{t+1}\right)$ back to SDF $\Rightarrow M_{t+1}=\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\left[\frac{U_{t+1}}{\mathcal{CE}_t(U_{t+1})}\right]^{\frac{1}{\psi}-\gamma}$

Epstein-Zin Utility (7/10)

SDF

$$M_{t+1} = \left\{ \delta \left(\frac{C_{t+1}}{C_t} \right)^{-1/\psi} \right\}^{\theta} \left\{ \frac{1}{1 + R_{w,t+1}} \right\}^{1-\theta}$$

Log SDF

$$m_{t+1} = \theta \left(\log \delta - \frac{1}{\psi} \Delta c_{t+1} \right) - (1-\theta) r_{w,t+1}$$

$$\tilde{m}_{t+1} = -\theta \left(\frac{\tilde{c}_{t+1}}{\psi} \right) - (1-\theta) \tilde{r}_{w,r+1}$$

Intuition: weighted average $(\theta, 1-\theta)$ of

1. Power utility (CCAPM) term
2. Wealth portfolio term
   - wealth portfolio as the "sufficient statistics" of future consumption news
3. When $\gamma = 1/\psi \Rightarrow \theta = 1$, converges to CCAPM.

Epstein-Zin Utility (8/10)

Use SDF to price riskfree asset

$$
\begin{aligned}
r_{f,t+1} &= -\mathrm{E}_t m_{t+1} - \frac{1}{2}\sigma_{mt}^2 \\
&= -\theta \log \delta + \frac{\theta}{\psi}\mathrm{E}_t \Delta c_{t+1} + (1-\theta)\mathrm{E}_t r_{w,t+1} \\
&\quad - \frac{\theta^2}{2\psi^2}\sigma_c^2 - \frac{(1-\theta)^2}{2}\sigma_w^2 - \frac{\theta(1-\theta)}{\psi}\sigma_{cw}
\end{aligned}
$$

Use SDF to price wealth portfolio

$$
\mathrm{E}_t r_{w,t+1} + \frac{1}{2}\sigma_{wt}^2 - r_{f,t+1} = -\sigma_{wmt} = \frac{\theta}{\psi}\sigma_{cw} + (1-\theta)\sigma_w^2
$$

Plugging wealth portfolio to get clean formula for riskfree rate

$$
r_{f,t+1} = \underbrace{-\log \delta + \frac{1}{\psi}\mathrm{E}_t \Delta c_{t+1}}_{\text{power utility (CCAPM)}} - \underbrace{\frac{1}{2}\theta \cdot \frac{1}{\psi^2}\sigma_c^2 - \frac{1}{2}(1-\theta)\sigma_w^2}_{\text{weighted average of risks}}
$$

Epstein-Zin Utility (9/10)

Use SDF to get risk premium

$$\mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma_i^2 - r_{f,t+1} = \frac{\theta}{\psi}\sigma_{ic} + (1-\theta)\sigma_{iw}$$

Relation between expected return and consumption growth

$$\mathrm{E}_t r_{i,t+1} = \frac{1}{\psi}\mathrm{E}_t \Delta c_{t+1} + \text{ constant }, \quad \tilde{r}_{i,t+1+j} = \frac{1}{\psi}\tilde{c}_{t+1+j}$$

- assume: jointly conditionally lognormal and homoskedastic
- $\rightarrow$ all second moments are constant
- $\rightarrow$ constant risk premium
- $\rightarrow$ return innovations only depends on consumption growth news

Epstein-Zin Utility (10/10)

Conclusion

- SDF:

$$m_{t+1} = \log\delta - \gamma\Delta c_{t+1} \qquad \text{power utility}$$

$$m_{t+1} = \theta\left(\log\delta - \frac{1}{\psi}\Delta c_{t+1}\right) - (1-\theta)r_{w,t+1} \qquad \text{EZ utility}$$

- risk free rate

$$r_{f,t+1} = -\log\delta + \gamma \mathrm{E}_t\Delta c_{t+1} - \frac{1}{2}\gamma^2\sigma_c^2 \qquad \text{power utility}$$

$$r_{f,t+1} = -\log\delta + \frac{1}{\psi}\mathrm{E}_t\Delta c_{t+1} - \frac{1}{2}\theta\frac{1}{\psi^2}\sigma_c^2 - \frac{1}{2}(1-\theta)\sigma_w^2 \qquad \text{EZ utility}$$

- risk premium

$$\mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma_{it}^2 - r_{f,t+1} = -\sigma_{imt} \qquad \text{power utility}$$

$$\mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma_i^2 - r_{f,t+1} = \frac{\theta}{\psi}\sigma_{ic} + (1-\theta)\sigma_{iw} \qquad \text{EZ utility}$$

- These results hold under lognormality and homoskedasticity assumptions. For more details, see Hansen and Singleton (1983), Attanasio and Weber (1989), and Vissing-Jørgensen (2002)

Wealth Portfolio Return Decomposition (1/3)

Apply the return decomposition to the wealth portfolio

$$\tilde{r}_{w,t+1} = (\mathrm{E}_{t+1} - \mathrm{E}_t) \sum_{j=0}^{\infty} \rho^j \Delta d_{w,t+1+j} - (\mathrm{E}_{t+1} - \mathrm{E}_t) \sum_{j=1}^{\infty} \rho^j r_{w,t+1+j}$$

Two "symmetric" model:

- extended CCAPM: consumption path is predictable?
- intertemporal CAPM: return is more precise?

Here we show:

- Substituting $r_{w,t+1+j}$ with Δc_{t+1+j} gives extended CCAPM
- Substituting Δc_{t+1+j} with $r_{w,t+1+j}$ gives intertemporal CAPM

Wealth Portfolio Return Decomposition (2/3)

Substituting $r_{w,t+1+j}$ with Δc_{t+1+j}

$$\tilde{r}_{w,t+1} = \tilde{c}_{t+1} + (\mathrm{E}_{t+1} - \mathrm{E}_t)\sum_{j=1}^{\infty}\rho^j \Delta c_{t+1+j} - (\mathrm{E}_{t+1} - \mathrm{E}_t)\sum_{j=1}^{\infty}\rho^j r_{w,t+1+j}$$

$$\xRightarrow{\tilde{r}_{w,t+1+j}=\frac{1}{\psi}\tilde{c}_{t+1+j}} \quad \tilde{r}_{w,t+1} = \tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right)(\mathrm{E}_{t+1} - \mathrm{E}_t)\sum_{j=1}^{\infty}\rho^j \Delta c_{t+1+j}$$

$$\xRightarrow{\tilde{g}_{t+1}\equiv\left(\mathrm{E}_{t+1}-\mathrm{E}_t\right)\sum_{j=1}^{\infty}\rho^j \Delta c_{t+1+j}} \quad \tilde{r}_{w,t+1} = \tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1}$$

- Can also do this for price-dividend (wealth-consumption) decomposition

$$w_t - c_t = (1 - 1/\psi)\mathrm{E}_t \sum_{j=0}^{\infty}\rho^j \Delta c_{t+1+j} + \text{ Constant}$$

Wealth Portfolio Return Decomposition (3/3)

Substituting Δc_{t+1+j} with $r_{w,t+1+j}$

$$\tilde{r}_{w,t+1} = \tilde{c}_{t+1} + (\mathrm{E}_{t+1} - \mathrm{E}_t) \sum_{j=1}^{\infty} \rho^j \Delta c_{t+1+j} - (\mathrm{E}_{t+1} - \mathrm{E}_t) \sum_{j=1}^{\infty} \rho^j r_{w,t+1+j}$$

$$\xrightarrow{\tilde{c}_{t+1+j} = \psi \tilde{r}_{w,t+1+j}} \quad \tilde{r}_{w,t+1} = \tilde{c}_{t+1} + (\psi - 1)(\mathrm{E}_{t+1} - \mathrm{E}_t) \sum_{j=1}^{\infty} \rho^j r_{w,t+1+j}$$

$$\xrightarrow{\tilde{h}_{t+1} \equiv (\mathrm{E}_{t+1} - \mathrm{E}_t) \sum_{j=1}^{\infty} \rho^j r_{w,t+1+j}} \quad \tilde{c}_{t+1} = \tilde{r}_{w,t+1} + (1 - \psi)\tilde{h}_{t+1}$$

- Can also do this for price-dividend (wealth-consumption) decomposition

$$w_t - c_t = (\psi - 1)\mathrm{E}_t \sum_{j=0}^{\infty} \rho^j r_{w,t+1+j} + \text{ Constant}$$

Extended Consumption CAPM (1/3)

SDF

$$\tilde{m}_{t+1} = -\theta\left(\frac{\tilde{c}_{t+1}}{\psi}\right) - (1-\theta)\tilde{r}_{w,r+1}$$

- Substituting r_w with Δc: $\tilde{r}_{w,t+1} = \tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1}$
- Check it! note $\theta = \frac{1-\gamma}{1-1/\psi}$.

$$\tilde{m}_{t+1} = \underbrace{-\gamma\tilde{c}_{t+1}}_{\text{power utility term}} - \underbrace{\left(\gamma - \frac{1}{\psi}\right)\tilde{g}_{t+1}}_{\text{News about future consumption}}$$

Risk premium

$$\mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma_i^2 - r_{f,t+1} = \gamma\sigma_{ic} + \left(\gamma - \frac{1}{\psi}\right)\sigma_{ig}$$

To derive BY model → Plug assumed processes in and take expectations.

Extended Consumption CAPM (2/3)

Risk premium on wealth portfolio

$$E_t r_{w,t+1} + \frac{1}{2}\sigma_w^2 - r_{f,t+1} = \gamma\sigma_{wc} + \left(\gamma - \frac{1}{\psi}\right)\sigma_{wg}$$

- Substituting r_w with Δc: $\tilde{r}_{w,t+1} = \tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1}$
- Assume $\mathrm{Cov}\left(\tilde{c}_{t+1}, \tilde{g}_{t+1}\right) = 0$

$$E_t r_{w,t+1} + \frac{1}{2}\sigma_w^2 - r_{f,t+1} = \gamma\sigma_c^2 + \left(\gamma - \frac{1}{\psi}\right)\left(1 - \frac{1}{\psi}\right)\sigma_g^2$$

Parameter restrictions on long run risk model (to settle risk premium puzzle)

1. Want shocks to consumption growth σ_g^2 to crease risk premium
2. $\gamma - \frac{1}{\psi} > 0 \rightarrow$ early resolution of uncertainty and averse positive σ_{ig}
3. $1 - \frac{1}{\psi} > 0(\psi > 1)$, so $\tilde{r}_{w,t+1}$ covaries positively with $\tilde{g}_{t+1}$.

Extended Consumption CAPM (3/3)

Calibration: long-ruin risk without stochastic volatility (BY Case I)

- Assume consumption growth process

$$x_{t+1} = \rho_x x_t + \varphi_e \sigma e_{t+1}$$
$$\Delta c_{t+1} = \mu + x_t + \sigma \eta_{t+1}, \qquad e, \eta, w \sim \text{ i.i.d. } N(0,1)$$

- Solving backwards of consumption growth process

$$\tilde{c}_{t+1} = (\mathrm{E}_{t+1} - \mathrm{E}_t)\, \Delta c_{t+1} = \sigma \eta_{t+1}$$
$$\tilde{c}_{t+1+j} = (\mathrm{E}_{t+1} - \mathrm{E}_t)\, \Delta c_{t+1+j} = \rho_x^{j-1} \varphi_e \sigma e_{t+1}$$

- Plugging consumption growth into SDF

$$\tilde{m}_{t+1} = -\gamma \tilde{c}_{t+1} - \left(\gamma - \frac{1}{\psi}\right) \sum_{j=1}^{\infty} \rho^j \tilde{c}_{t+1+j} = -\gamma \sigma \eta_{t+1} - \left(\gamma - \frac{1}{\psi}\right) \frac{\rho \varphi_e}{1 - \rho \rho_x} \sigma e_{t+1}$$

- Pricing wealth portfolio to get equity premium

$$\mathrm{E}_t r_{w,t+1} + \frac{1}{2}\sigma_w^2 - r_{f,t+1} = -\,\mathrm{Cov}\left(\tilde{m}_{t+1}, \tilde{r}_{w,t+1}\right)$$
$$= \gamma \sigma^2 + \left(\gamma - \tfrac{1}{\psi}\right)\left(1 - \tfrac{1}{\psi}\right)\left(\tfrac{\rho \varphi_e}{1 - \rho \rho_x}\right)^2 \sigma^2$$

Intertemporal CCAPM (1/3)

SDF

$$\tilde{m}_{t+1} = -\theta \left(\frac{\tilde{c}_{t+1}}{\psi} \right) - (1-\theta)\tilde{r}_{w,r+1}$$

- Substituting Δc with r_w: $\tilde{c}_{t+1} = \tilde{r}_{w,t+1} + (1-\psi)\tilde{h}_{t+1}$
- Check it! note $\theta = \frac{1-\gamma}{1-1/\psi}$.

$$\tilde{m}_{t+1} = \underbrace{-\gamma\tilde{r}_{w,t+1}}_{\text{power utility term}} - \underbrace{(\gamma-1)\tilde{h}_{t+1}}_{\text{Intertemporal hedging term}}$$

Risk premium

$$\mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma_i^2 - r_{f,t+1} = \gamma\sigma_{iw} + (\gamma-1)\,\sigma_{ih}$$

- when $\gamma = 1$, it's myopic portfolio choice $\rightarrow$ standard CAPM

To derive CV "good/bad betas" $\rightarrow$ Apply return decomposition to wealth portfolio then replace σ_{iw}, σ_{ih}

Intertemporal CCAPM (2/3)

CV two beta model

- Return decomposition of wealth portfolio

$$\tilde{r}_{w,t+1} = \sum_{j=0}^{\infty} \rho^j \tilde{c}_{t+1+j} - \sum_{j=1}^{\infty} \rho^j \tilde{r}_{w,t+1+j} \equiv N_{CF,t+1} - N_{DR,t+1}$$

- Plugging back to SDF $\tilde{m}_{t+1} = -\gamma \tilde{r}_{w,t+1} - (\gamma - 1)\tilde{h}_{t+1}$:

$$\tilde{m}_{t+1} = -\gamma N_{CF,t+1} + N_{DR,t+1}$$

- Risk premium

$$E_t r_{i,t+1} + \frac{1}{2}\sigma_i^2 - r_{f,t+1} = \gamma \sigma_{iCF} + \sigma_{i-DR}$$
$$= \gamma \sigma_w^2 \underbrace{\beta_{iCF}}_{\text{bad beta}} + \sigma_w^2 \underbrace{\beta_{i-DR}}_{\text{good beta}}$$

1. good beta: news about the market's discount rates.
2. bad beta: news about the market's future cash flows (higher price of risk hence "bad").

Intertemporal CCAPM (3/3)

Calibration: CV good beta and bad beta

1. Assume process: state vector z_t follows VAR(1), the first element is $r_{w,t}$
   - Choosing variables in the z_t not trivial, typical: PE ratio, value spread, default spread, term spread

$$z_{t+1} = \bar{z} + \Gamma\left(z_t - \bar{z}\right) + u_{t+1}$$

2. Solving backwards, use state vector to express $\tilde{r}_w$

$$\tilde{r}_{w,t+1} = e_1' u_{t+1}, \quad \tilde{r}_{w,t+1+j} = e_1' \Gamma^j u_{t+1}$$

3. Use state vector to express $N_{DR,t+1}$ and $N_{CF,t+1}$

$$N_{DR,t+1} = \left(\mathrm{E}_{t+1} - \mathrm{E}_t\right) \sum_{j=1}^{\infty} \rho^j r_{w,t+1+j} = e_1' \lambda u_{t+1}$$

$$N_{CF,t+1} = \tilde{r}_{w,t+1} + N_{DR,t+1} = \left(e_1' + e_1' \lambda\right) u_{t+1}, \quad \text{where } \lambda \equiv \rho\Gamma(I - \rho\Gamma)^{-1}$$

4. Then plgging back to SDF, use state vector to express SDF

$$\tilde{m}_{t+1} = -\gamma N_{CF,t+1} + N_{DR,t+1} = \widetilde{m}_{t+1} = -\gamma\left(e_1' + e_1' \lambda\right) u_{t+1} + e_1' \lambda u_{t+1}$$

5. Then can use such SDF to do pricing – Done!

Risk Premium under Three Models

CCAPM

$$\mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma_i^2 - r_{f,t+1} = \gamma \sigma_{ic}$$

- risk premium = compensation for risk
- price of risk γ and quantity of risk σ_{ic}
- $\sigma_{ic} > 0$: stock (high expected return)
- $\sigma_{ic} < 0$: insurance (low expected return)
- $\sigma_{ic} \to 0$: risk-free asset

Risk Premium under Three Models

Extended CAPM

$$\mathrm{E}_t r_{i,t+1} + \frac{1}{2}\sigma_i^2 - r_{f,t+1} = \gamma\sigma_{ic} + \left(\gamma - \frac{1}{\psi}\right)\sigma_{ig}$$

- Two types of risk consideration: next period consumption and future consumption growth
- Assume people prefer early resolution of risk: $\gamma > 1/\psi$
- Then demand for positive premium for the future consumption growth
- Campbell section 6.4 & 6.5

Risk Premium under Three Models

Intertemporal CAPM

$$E_t r_{i,t+1} + \frac{1}{2}\sigma_i^2 - r_{f,t+1} = \gamma\sigma_{iCF} + \sigma_{i-DR}$$
$$= \gamma\sigma_w^2 \underbrace{\beta_{iCF}}_{\text{bad beta}} + \sigma_w^2 \underbrace{\beta_{i-DR}}_{\text{good beta}}$$

- Two types of risk consideration: news about cash flow and future discount rate
- "bad beta": cash flow news has higher risk price $\gamma \rightarrow$ investors care more about cash flow news than discount rate news
- Campbell section 9.3