Asset Pricing Review Session 5

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Outlines for review sessions

First Part (Static Asset Pricing)

1. Choice under uncertainty
2. Static portfolio choice
3. Static asset pricing
4. Stochastic discount factor

Second Part (Intertemporal Asset Pricing)

1. Present value relations
2. Long run risk (BY)
3. Intertemporal CAPM (CV)
4. Rare Disaster (Martin)
5. Stochastic volatility (BKY and CGPT)
6. Intertemporal portfolio choice
7. Term structure & bond pricing

Last Session

1. Refinements on logs and approximations
   - Risk premium, pricing equation, portfolio choice
   - Campbell Shiller approximation and decompositions
2. Examples: two utility functions
   - Power utility and Epstein Zin utility
   - SDF, riskfree rate, and risk premium
3. Epstein Zin with return decompositions
   - Extended consumption CAPM (long run risk)
   - Intertemporal CAPM (“good beta and bad beta”)

Today's Outline

Today

1. Rare disaster model: Martin (2013).
2. Return decompositions with stochastic volatility.
3. Extended consumption CAPM + stochastic volatility.
4. Intertemporal CAPM + stochastic volatility.

Keys

- Break lognormality assumption → Martin (2013)
- Break homoskedasticity
  - Add the stochastic volatility to these models
  - Solve full models with time series specifications

Rare Disaster by Martin (2013)

- Three "tricks"
  1. Solving pricing equation recursively
     - Link prices to future cash flows (dividends)
  2. Pricing levered equity
     - Link dividends to consumption $D_t = C_t^{\lambda}, d_t = \lambda c_t$.
     - Riskfree asset: $\lambda = 0$; Wealth portfolio $\lambda = 1$.
  3. Cumulant generating function
     - A general expression for expectations of exponentials

- Model Setup
  - Power utility, time discount factor $\delta = e^{-r^*}$, risk aversion γ
  - Consumption growth is i.i.d. $G = c_{t+1} - c_t$
  - No lognormal assumption!

Rare Disaster by Martin (2013)

Step 1: Solving pricing equation recursively

$$P_t = \mathrm{E}_t\left[M_{t+1}\left(P_{t+1} + D_{t+1}\right)\right]$$

$$\cdots$$

$$P_{t+j} = \mathrm{E}_{t+j}\left[M_{t+j+1}\left(P_{t+j+1} + D_{t+j+1}\right)\right]$$

Solving forward to infinity

$$P_t = \mathrm{E}_t\left[\sum_{j=1}^{\infty}\left(\prod_{k=1}^{j} M_{t+k}\right) D_{t+j}\right] + \underbrace{\lim_{j\to\infty} \mathrm{E}_t\left[\left(\prod_{k=1}^{j} M_{t+k}\right) P_{t+j}\right]}_{=0,\text{ no-bubble condition}}$$

Rare Disaster by Martin (2013)

Step 2: Pricing levered equity

$$P_t = \mathrm{E}_t \left[\sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} M_{t+k} \right) D_{t+j} \right]$$

- For power utility, $M_{t+1} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$

$$P_t = \mathrm{E}_t \left[\sum_{j=1}^{\infty} \delta^j \left(\frac{C_{t+j}}{C_t} \right)^{-\gamma} D_{t+j} \right]$$

- Time discount factor $\delta = e^{-r^*}$, dividends $D_t = C_t^{\lambda}$, consumption growth rate $G_{t+1} = c_{t+1} - c_t$ i.i.d.

$$P_t = \mathrm{E}_t \left[\sum_{j=1}^{\infty} e^{-jr^*} \left(\frac{C_{t+j}}{C_t} \right)^{-\gamma} C_{t+j}^{\lambda} \right] = D_t \mathrm{E}_t \left[\sum_{j=1}^{\infty} e^{-jr^*} \left(\frac{C_{t+j}}{C_t} \right)^{\lambda-\gamma} \right]$$

$$= D_t \sum_{j=1}^{\infty} e^{-jr^*} \mathrm{E}_t \left[\prod_{k=1}^{j} e^{(\lambda-\gamma)G} \right] = D_t \sum_{j=1}^{\infty} e^{-jr^*} \left(E \left[e^{(\lambda-\gamma)G} \right] \right)^j$$

Rare Disaster by Martin (2013)

Step 3: Cumulant generating function (CGF)

$$P_t = D_t \sum_{j=1}^{\infty} e^{-jr^*} \left(E\left[e^{(\lambda-\gamma)G}\right]\right)^j$$

$$= D_t \sum_{j=1}^{\infty} e^{-j\left(r^* - c(\lambda-\gamma)\right)}$$

- Constant dividend-to-price ratio

$$\frac{D_t}{P_t} + 1 = e^{r^* - c(\lambda-\gamma)} \Longrightarrow \log\left(\frac{D_t}{P_t} + 1\right) = dp(\lambda) = r^* - \boldsymbol{c}(\lambda - \gamma)$$

Aside: cumulant generating function

$$\boldsymbol{c}(\theta) \equiv \log E e^{\theta G}, \quad \boldsymbol{c}(\theta) \equiv \sum_{n=1}^{\infty} \frac{\kappa_n \theta^n}{n!}$$

$$\Longrightarrow \quad \mathrm{E} e^{\theta G} = e^{c(\theta)}, \quad \mathrm{E} e^{(\lambda-\gamma)G} = e^{c(\lambda-\gamma)}$$

Aside: cumulant generating function

$$\boldsymbol{c}(\theta) \equiv \sum_{n=1}^{\infty} \frac{\kappa_n \theta^n}{n!}$$

- κ_1 is the mean of log consumption growth.
- κ_2 is the variance σ^2.
- κ_3/σ^3 is the skewness
- κ_4/σ^4 is the excess kurtosis, and so forth.
- when log consumption growth is normal: all cumulants above the second are zero consumption growth.

Rare Disaster by Martin (2013)

From return identity to equity premium

$$1 + R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{P_{t+1}}{P_t}\left(1 + \frac{D_{t+1}}{P_{t+1}}\right)$$

- show $E_t[P_{t+1}/P_t] = e^{c(\lambda)}$?
- equity premium

$$\begin{aligned}
\log(1 + ER_{t+1}) &= \mathrm{er}(\lambda) = r^* + \boldsymbol{c}(\lambda) - \boldsymbol{c}(\lambda - \gamma) \\
r_f &= \mathrm{er}(0) = r^* - \boldsymbol{c}(-\gamma) \\
r_w &= \mathrm{er}(1) = r^* + \boldsymbol{c}(1) - \boldsymbol{c}(1 - \gamma) \\
eqp &= r_w - r_f = \boldsymbol{c}(1) + \boldsymbol{c}(-\gamma) - \boldsymbol{c}(1 - \gamma) \\
&= \sum_{n=1}^{\infty} \frac{\kappa_n}{n'} \left[1 + (-\gamma)^n - (1 - \gamma)^n\right]
\end{aligned}$$

- when $n = 1$, $eqp = 0$ (risk free rate)
- when $n = 2$, $eqp = \gamma\sigma^2$ (i.i.d. lognormal consumption growth).

Rare Disaster by Martin (2013)

Martin (2013): a general method for solving different types of model

- Can be extended to the cases of
  1. Log consumption not proportional to log dividends (not levered equity)
     - Need bivariate CGF
$$\boldsymbol{c}(\theta_1, \theta_2) \equiv \log \mathrm{E} e^{\theta_1 x_1 + \theta_2 x_2}$$
  2. Non i.i.d . consumption growth
     - Need conditional (time varying) bivariate CGF
$$c_t(\theta_1, \theta_2) \equiv \log \mathrm{E}_t e^{\theta_1 x_{1,t+1} + \theta_2 x_{2,t+1}}$$
  3. Epstein Zin utility
     - Need some identities (manipulations) of the wealth portfolio
- Barro (2006, 2009): special cases with jumps
- For more details, see Martin (2013) Section 4!

Return Decomposition without Stochastic Volatility

- Apply the return decomposition to the wealth portfolio

$$\tilde{r}_{w,t+1} = \tilde{c}_{t+1} + \sum_{j=1}^{\infty} \rho^j \tilde{c}_{t+1+j} - \sum_{j=1}^{\infty} \rho^j \tilde{r}_{w,t+1+j}$$

  - This identity holds no matter the volatility is time varying or not
- Substituting $r_{w,t+1+j}$ with Δc_{t+1+j} gives extended consumption CAPM (CCAPM)
- Substituting Δc_{t+1+j} with $r_{w,t+1+j}$ gives intertemporal CAPM (ICAPM)
- How to substitute?

$$\tilde{r}_{w,t+1+j} = \frac{1}{\psi} \tilde{c}_{t+1+j}$$

  - This relation relies on constant (co)variances assumption
  - Now: drive this under stochastic volatility.

Return Decomposition with Stochastic Volatility

- Starting from the pricing equation

$$\mathrm{E}_t r_{w,t+1} = -\mathrm{E}_t m_{t+1} - \frac{1}{2}\operatorname{Var}_t\left(m_{t+1} + r_{w,t+1}\right)$$

- Plugging into E-Z utility SDF $m_{t+1} = \theta \log \delta - \frac{\theta}{\psi}\Delta c_{t+1} - (1-\theta) r_{w,t+1}$

$$\begin{aligned}\mathrm{E}_t r_{w,t+1} &= -\theta \log \delta + \frac{\theta}{\psi}\mathrm{E}_t \Delta c_{t+1} + (1-\theta)\mathrm{E}_t r_{w,t+1} - \frac{1}{2}\operatorname{Var}_t\left(m_{t+1} + r_{w,t+1}\right) \\ &= -\log \delta + \frac{1}{\psi}\mathrm{E}_t \Delta c_{t+1} - \frac{1}{2\theta} V_t\end{aligned}$$

- Then the relation between wealth portfolio and consumption growth

$$\tilde{r}_{w,t+1+j} = \frac{1}{\psi}\tilde{c}_{t+1+j} - \frac{1}{2\theta}\tilde{V}_{t+j}$$

Return Decomposition with Stochastic Volatility

Substitute $r_{w,t+1+j}$ with Δc_{t+1+j}

$$\tilde{r}_{w,t+1} = \tilde{c}_{t+1} + \sum_{j=1}^{\infty} \rho^j \tilde{c}_{t+1+j} - \sum_{j=1}^{\infty} \rho^j \tilde{r}_{w,t+1+j}$$

$$\xrightarrow{\tilde{r}_{w,t+1+j} = \frac{1}{\psi}\tilde{c}_{t+1+j} - \frac{1}{2\theta}\tilde{V}_{t+j}} \tilde{r}_{w,t+1} = \tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right) \sum_{j=1}^{\infty} \rho^j \tilde{c}_{t+1+j} + \frac{1}{2\theta} \underbrace{\sum_{j=1}^{\infty} \rho^j \tilde{V}_{t+j}}_{\equiv N_{RISK,t+1}}$$

Substitute Δc_{t+1+j} **with** $r_{w,t+1+j}$

$$\tilde{c}_{t+1} = \tilde{r}_{w,t+1} - \sum_{j=1}^{\infty} \rho^j \tilde{c}_{t+1+j} + \sum_{j=1}^{\infty} \rho^j \tilde{r}_{w,t+1+j}$$

$$\xrightarrow{\tilde{c}_{t+1+j} = \psi \tilde{r}_{w,t+1+j} + \frac{\psi}{2\theta}\tilde{V}_{t+j}} \tilde{c}_{t+1} = \tilde{r}_{w,t+1} + (1 - \psi) \sum_{j=1}^{\infty} \rho^j \tilde{r}_{w,t+1+j} - \frac{\psi}{2\theta} \underbrace{\sum_{j=1}^{\infty} \rho^j \tilde{V}_{t+j}}_{\equiv N_{RISK,t+1}}$$

Extended Consumption CAPM

SDF

$$\tilde{m}_{t+1} = -\theta \left(\frac{\tilde{c}_{t+1}}{\psi} \right) - (1-\theta)\tilde{r}_{w,r+1}$$

- Substitute $r_{w,t+1+j}$ with Δc_{t+1+j}

$$\tilde{r}_{w,t+1} = \tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right) \sum_{j=1}^{\infty} \rho^j \tilde{c}_{t+1+j} + \frac{1}{2\theta} N_{RISK,t+1}$$

$$\Rightarrow \tilde{m}_{t+1} = -\gamma \tilde{c}_{t+1} - \left(\gamma - \frac{1}{\psi}\right) \sum_{j=1}^{\infty} \rho^j \tilde{c}_{t+1+j} - \frac{1-\theta}{2\theta} \underbrace{N_{RISK,t+1}}_{\text{News about risk}}$$

- Not finished yet – $N_{RISK,t+1}$ is a function of SDF:

$$N_{RISK,t+1} = (\mathrm{E}_{t+1} - \mathrm{E}_t) \sum_{j=1}^{\infty} \rho^j \operatorname{Var}_{t+j} \left(m_{t+1+j} + r_{w,t+1+j} \right)$$

- Key: solve $\tilde{m}_{t+1}$ by the relation $\tilde{m}_{t+1} = f\left(\tilde{m}_{t+1}\right)$
- If the specifications are linear (e.g. AR(1), VAR)
  - One state variable drives all first (second) moments
  - Guess and verify state variables drive moments in a linear way

Extended Consumption CAPM

Example: long run risk with stochastic volatility (BY Case II/BKY)

$$x_{t+1} = \rho_x x_t + \varphi_e \sigma_t e_{t+1}$$
$$\Delta c_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}$$
$$\sigma^2_{t+1} = \sigma^2 + v\left(\sigma_t^2 - \sigma^2\right) + \sigma_w w_{t+1}$$
$$e, \eta, w \sim \text{ i.i.d. } N(0,1)$$

1. Solving backwards

$$\tilde{c}_{t+1} = \left(\mathrm{E}_{t+1} - \mathrm{E}_t\right) \Delta c_{t+1} = \sigma_t \eta_{t+1}$$
$$\tilde{c}_{t+1+j} = \left(\mathrm{E}_{t+1} - \mathrm{E}_t\right) \Delta c_{t+1+j} = \rho_x^{j-1} \varphi_e \sigma_t e_{t+1}$$
$$\tilde{V}_{t+j} = \left(\mathrm{E}_{t+1} - \mathrm{E}_t\right) V_{t+j} = \omega v^{j-1} \sigma_w w_{t+1}$$

Extended Consumption CAPM

1. Solving backwards

$$\tilde{c}_{t+1} = \left(\mathrm{E}_{t+1} - \mathrm{E}_t\right) \Delta c_{t+1} = \sigma_t \eta_{t+1}$$
$$\tilde{c}_{t+1+j} = \left(\mathrm{E}_{t+1} - \mathrm{E}_t\right) \Delta c_{t+1+j} = \rho_x^{j-1} \varphi_e \sigma_t e_{t+1}$$
$$\tilde{V}_{t+j} = \left(\mathrm{E}_{t+1} - \mathrm{E}_t\right) V_{t+j} = \omega v^{j-1} \sigma_w w_{t+1}$$

2. Guess: $V_{t+j} = \mathrm{Var}_{t+j}\left(m_{t+1+j} + r_{w,t+1+j}\right) = \omega \sigma_{t+j}^2 + \eta$, then the SDF:

$$\begin{aligned}
\tilde{m}_{t+1} &= -\gamma \tilde{c}_{t+1} - \left(\gamma - \frac{1}{\psi}\right) \sum_{j=1}^{\infty} \rho^j \tilde{c}_{t+1+j} - \frac{1-\theta}{2\theta} N_{RISK,t+1} \\
&= -\gamma \sigma_t \eta_{t+1} - \left(\gamma - \frac{1}{\psi}\right) \sum_{j=1}^{\infty} \rho^j \rho_x^{j-1} \varphi_e \sigma_t e_{t+1} - \frac{1-\theta}{2\theta} \sum_{j=1}^{\infty} \rho^j \omega \nu^{j-1} \sigma_w w_{t+1} \\
&= -\gamma \sigma_t \eta_{t+1} - \left(\gamma - \frac{1}{\psi}\right) \frac{\rho \varphi_e}{1 - \rho \rho_x} \sigma_t e_{t+1} - \frac{1-\theta}{2\theta} \frac{\omega \sigma_w \rho}{1 - \rho v} w_{t+1}
\end{aligned}$$

Extended Consumption CAPM

3. Verify: $V_t = \text{Var}_t\left(m_{t+1} + r_{w,t+1}\right) = \omega\sigma_t^2 + \eta$

$$V_t = \text{Var}_t\left(m_{t+1} + r_{w,t+1}\right) = \text{Var}_t\left(\tilde{m}_{t+1} + \tilde{r}_{w,t+1}\right)$$

$$= \text{Var}_t\left[(1-\gamma)\sum_{j=0}^{\infty}\rho^j\tilde{c}_{t+1+j} + \frac{1}{2}N_{RISK,t+1}\right]$$

- Plugging into time-series specifications

$$\tilde{c}_{t+1} = \Delta c_{t+1} = \sigma_t\eta_{t+1}, \quad \tilde{c}_{t+1+j} = \rho_x^{j-1}\varphi_e\sigma_t e_{t+1}$$

$$\tilde{V}_{t+j} = \omega v^{j-1}\sigma_w w_{t+1}, \quad e,\eta,w \sim \text{ i.i.d. } N(0,1)$$

- Pin down ω by equating coefficients

$$V_t = \text{Var}_t\left[(1-\gamma)\sigma_t\eta_{t+1} + (1-\gamma)\frac{\rho\varphi_e}{1-\rho\rho_x}\sigma_t e_{t+1} + \frac{1}{2}\frac{\omega\sigma_w\rho}{1-\rho v}w_{t+1}\right]$$

$$= \omega\sigma_t^2 + \eta = (1-\gamma)^2\sigma_t^2 + (1-\gamma)^2\left(\frac{\rho\varphi_e}{1-\rho\rho_x}\right)^2\sigma_t^2 + \frac{1}{4}\omega^2\left(\frac{\sigma_w\rho}{1-\rho v}\right)^2$$

$$= \omega\sigma_t^2 + \eta$$

- $\omega = (1-\gamma)^2\left[1 + \left(\frac{\rho\varphi_e}{1-\rho\rho_x}\right)^2\right]$

Extended Consumption CAPM

4. Plug ω back into SDF

$$\begin{aligned}\tilde{m}_{t+1} =& -\gamma\sigma_t\eta_{t+1} - \left(\gamma - \frac{1}{\psi}\right)\frac{\rho\varphi_e}{1-\rho\rho_x}\sigma_t e_{t+1} \\ & - \frac{1}{2}\left(\gamma - \frac{1}{\psi}\right)(1-\gamma)\left[1 + \left(\frac{\rho\varphi_e}{1-\rho\rho_x}\right)^2\right]\frac{\rho}{1-\rho v}\sigma_w w_{t+1} \\ =& \lambda_{m,\eta}\sigma_t\eta_{t+1} - \lambda_{m,e}\sigma_t e_{t+1} - \lambda_{m,w}\sigma_w w_{t+1}\end{aligned}$$

- $\lambda_{m,\eta}\sigma_t\eta_{t+1} - \lambda_{m,e}\sigma_t e_{t+1}$: BY without stochastic volatility
- $\lambda_{m,w} = 0$ if $\gamma = \frac{1}{\psi}$ (power utility) or $\gamma = 1$ (myopic portfolio choice)

5. Price wealth portfolio to get equity premium

$$\begin{aligned}\mathrm{E}_t r_{w,t+1} + \frac{1}{2}\sigma_w^2 - r_{f,t+1} =& -\operatorname{Cov}\left(\tilde{m}_{t+1}, \tilde{r}_{w,t+1}\right) \\ =& \gamma\sigma_t^2 + \left(\gamma - \frac{1}{\psi}\right)\left(1 - \frac{1}{\psi}\right)\left(\frac{\rho\varphi_e}{1-\rho\rho_x}\right)^2\sigma_t^2 \\ & + \frac{1}{4}\left(\gamma - \frac{1}{\psi}\right)\left(1 - \frac{1}{\psi}\right)(1-\gamma)^2\left[1 + \left(\frac{\rho\varphi_e}{1-\rho\rho_x}\right)^2\right]^2\left(\frac{\rho\sigma_w}{1-\rho v}\right)^2\end{aligned}$$

Intertemporal CAPM

SDF

$$\tilde{m}_{t+1} = -\theta\left(\frac{\tilde{c}_{t+1}}{\psi}\right) - (1-\theta)\tilde{r}_{w,r+1}$$

- Substitute Δc_{t+1+j} with $r_{w,t+1+j}$

$$\tilde{c}_{t+1} = \tilde{r}_{w,t+1} + (1-\psi)\sum_{j=1}^{\infty}\rho^j \tilde{r}_{w,t+1+j} - \frac{\psi}{2\theta}N_{RISK,t+1}$$

$$\Rightarrow \tilde{m}_{t+1} = -\gamma\tilde{r}_{w,t+1} - (\gamma-1)\sum_{j=1}^{\infty}\rho^j \tilde{r}_{w,t+1+j} + \frac{1}{2}\underbrace{N_{RISK,t+1}}_{\text{News about risk}}$$

$$= -\gamma N_{CF,t+1} + N_{DR,t+1} + \frac{1}{2}N_{RISK,t+1}$$

- Not finished yet – $N_{RISK,t+1}$ is a function of SDF:

$$N_{RISK,t+1} = (\mathrm{E}_{t+1} - \mathrm{E}_t)\sum_{j=1}^{\infty}\rho^j \operatorname{Var}_{t+j}\left(m_{t+1+j} + r_{w,t+1+j}\right)$$

- Key: solve $\tilde{m}_{t+1}$ by the relation $\tilde{m}_{t+1} = f\left(\tilde{m}_{t+1}\right)$ – guess and verify

Intertemporal CAPM

1. Guess $V_{t+j} = \mathrm{Var}_{t+j}\left(m_{t+1+j} + r_{w,t+1+j}\right) = \omega\sigma^2_{t+j}$. Then

$$N_{RISK,t+1} = \left(\mathrm{E}_{t+1} - \mathrm{E}_t\right)\sum_{j=1}^{\infty}\rho^j\,\mathrm{Var}_{t+j}\left(m_{t+1+j} + r_{w,t+1+j}\right)$$

$$= \omega\left(\mathrm{E}_{t+1} - \mathrm{E}_t\right)\sum_{j=1}^{\infty}\rho^j\sigma^2_{t+j} \equiv \omega N_{V,t+1}$$

2. Verify – to pins down ω

$$V_t = \omega\sigma_t^2 = \mathrm{Var}_t\left[(1-\gamma)N_{CF,t+1} + \frac{1}{2}N_{RISK,t+1}\right]$$

$$= \mathrm{Var}_t\left[(1-\gamma)N_{CF,t+1} + \frac{1}{2}\omega N_{V,t+1}\right]$$

$$= (1-\gamma)^2\,\mathrm{Var}_t\left[N_{CF,t+1}\right] + \omega^2\frac{1}{4}\,\mathrm{Var}_t\left[N_{V,t+1}\right]$$

$$+\,\omega(1-\gamma)\,\mathrm{Cov}_t\left[N_{CF,t+1}, N_{V,t+1}\right]$$

3. Plug ω back to SDF

$$\tilde{m}_{t+1} = -\gamma N_{CF,t+1} + N_{DR,t+1} + \frac{1}{2}\omega N_{V,t+1}$$

Intertemporal CAPM

Example: three-beta model (CGPT)

1. Assume state vector z_t follows AR(1): $z_{t+1} = \bar{z} + \Gamma\left(z_t - \bar{z}\right) + \sigma_t u_{t+1}$
   - The first and second element are $r_{w,t}$ and σ_t^2.
   - u_{t+1} has constant variance covariance matrix Σ
2. Solving backwards

$$\tilde{r}_{w,t+1} = e_1' \sigma_t u_{t+1}, \quad \tilde{r}_{w,t+1+j} = e_1' \Gamma^j \sigma_t u_{t+1}, \quad \tilde{\sigma}^2{}_{t+j} = e_2' \Gamma^{j-1} \sigma_t u_{t+1}$$

3. Express news terms

$$N_{DR,t+1} = \left(\mathrm{E}_{t+1} - \mathrm{E}_t\right) \sum_{j=1}^{\infty} \rho^j r_{w,t+1+j} = e_1' \lambda \sigma_t u_{t+1}$$

$$N_{CF,t+1} = \tilde{r}_{w,t+1} + N_{DR,t+1} = \left(e_1' + e_1' \lambda\right) \sigma_t u_{t+1}$$

$$N_{RISK,t+1} = \omega N_{V,t+1} = \omega\left(\mathrm{E}_{t+1} - \mathrm{E}_t\right) \sum_{j=1}^{\infty} \rho^j \sigma_{t+j}^2 = \omega e_2' \rho (I - \rho\Gamma)^{-1} \sigma_t u_{t+1}$$

Intertemporal CAPM

Example: three-beta model (CGPT)

4. Plugging into SDF

$$\begin{aligned}\tilde{m}_{t+1} &= -\gamma N_{CF,t+1} + N_{DR,t+1} + \frac{1}{2} N_{RISK,t+1} \\ &= -\gamma \left(e_1' + e_1'\lambda\right)\sigma_t u_{t+1} + e_1'\lambda\sigma_t u_{t+1} + \frac{1}{2}\omega e_2'\rho(I-\rho\Gamma)^{-1}\sigma_t u_{t+1}\end{aligned}$$

5. Guess $V_{t+j} = \omega\sigma_{t+j}^2$. Pin down ω through guess-and-verify

$$\begin{aligned}V_t = \omega\sigma_t^2 &= \mathrm{Var}_t\left[(1-\gamma)N_{CF,t+1} + \frac{1}{2}N_{RISK,t+1}\right] \\ &= \mathrm{Var}_t\left[(1-\gamma)N_{CF,t+1} + \frac{1}{2}\omega N_{V,t+1}\right] \\ &= (1-\gamma)^2\,\mathrm{Var}_t\left[N_{CF,t+1}\right] + \omega^2\frac{1}{4}\,\mathrm{Var}_t\left[N_{V,t+1}\right] \\ &\quad + \omega(1-\gamma)\,\mathrm{Cov}_t\left[N_{CF,t+1}, N_{V,t+1}\right]\end{aligned}$$

that is,

$$0 = \omega^2\frac{1}{4}x_V\Sigma x_V' - \omega\left(1-(1-\gamma)x_{CF}\Sigma x_V'\right) + (1-\gamma)^2 x_{CF}\Sigma x_{CF}'$$

where $\lambda \equiv \rho\Gamma(I-\rho\Gamma)^{-1}$; $x_{CF} \equiv \left(e_1' + e_1'\lambda\right)$; $x_V \equiv e_2'\rho(I-\rho\Gamma)^{-1}$

Overview

Finance studies have three main dimensions

1. risk – choice under uncertainty (risk taking behavior)
2. time – save, borrow, invest; static v.s. intertemporal
3. information – asymmetric information (moral hazard, adverse selection)

We have two blocks this semesters:

1. Part 1: finance foundation. (static asset pricing)
2. Part 2: Intertemporal asset pricing (consumption based asset pricing)

Static Asset Pricing

1. Choice under uncertainty:
   - Utility theory, especially expected utility.
   - Risk aversion: concavity of the utility function → precautionary saving

$$R(W) = -\frac{u''(W)}{u'(W)}W, \quad A(W) = -\frac{u''(W)}{u'(W)}$$

   - Risk premium and certainty equivalence

$$u(W-\pi) = Eu(W+\tilde{x})$$
$$u\left(W+C^e\right) = Eu(W+\mu+\tilde{x})$$

   - Intertemporal marginal rate of substitution (EIS) = SDF

$$\text{IMRS}_{t+1,t} = \frac{\partial U_t/\partial C_{t+1}}{\partial U_t/\partial C_t}$$

   - Elasticity of intertemporal substitution (EIS) → consumption smooth incentive

$$\text{EIS} = -\frac{\partial \log\left(\frac{C_{t+1}}{C_t}\right)}{\partial \log \text{IMRS}_{t+1,t}}$$

Static Asset Pricing

2. Static portfolio choice
- Portfolio choice of mean-variance efficient maximizers
- 4 scenarios: principle of participation, small risk case, CARA-normal case, CRRA-lognormal case.

3. Static asset pricing
- CAPM: perfect risk sharing env, find mean-variance efficient portfolio under eqm argument

$$\bar{R}_i - R_f = \frac{\text{Cov}\left(R_i, R_m\right)}{\text{Var}\left(R_m\right)}\left(\bar{R}_m - R_f\right) = \beta_{im}\left(\bar{R}_m - R_f\right)$$

- Arbitrage Pricing Theory: assume a factor structure, reduce dimensions, 1-factor model = CAPM

$$R_{it} - R_f = \mu_i + \sum_{k=1}^{K} \beta_{ik} F_{kt} + \epsilon_{it}, \quad \mu_i = \sum_{k=1}^{K} \beta_{ik} \lambda_k.$$

$$\text{where } \lambda_k = E\left[R_{kt} - R_f\right] \text{ and } F_{kt} = R_{kt} - R_f - \lambda_k$$

Static Asset Pricing

4. Stochastic discount factor
   - intuition: "stochastic" + "discount factor"

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}, \quad E_t[M_{t+1}R_{t+1}] = 1.$$

   - fundamental asset pricing equation and beta representation,
   - formal definition, existence, law of one price, absence of arbitrage
   - SDF in complete market (contingent claim) vs incomplete market.
   - properties: volatility bounds

$$\left|\frac{\sigma(M)}{\mathrm{E}M}\right| \geq \underbrace{\left|\frac{\mathrm{E}R_i - R_f}{\sigma\left(R_i - R_f\right)}\right|}_{\text{Sharpe Ratio}}$$

   - derive CAPM via SDF: utility + eqm argument → find SDF → plug into pricing equation
   - construct SDF of factor model: factor structure argument → construct SDF with factors

Intermporal Asset Pricing

1. Present value relations
   - asset pricing facts/puzzles, Gordon Growth Model

$$P = \frac{D}{r-g}$$

   - Consumption-based CAPM: intuition, fails to resolve asset pricing puzzles

$$r_{f,t+1} = -\log\delta + \gamma \mathrm{E}_t \Delta c_{t+1} - \frac{1}{2}\gamma^2\sigma_c^2$$

2. Extended CCAPM and Intertemporal CCAPM model
   - Epstein-Zin utility: separate EIS and RA parameters, SDF construction with wealth portfolio
   - Return decomposition → replace c with r or replace r with c
   - Express SDF in terms of c or r and solve the model using backward induction
   - Add stochastic volatility, N_{risk}
3. Rare Disaster model: break lognormal assumption
   - Barro model, "disaster risk setting"
   - Martin (2013) technique: cumulant generating function

Review for Session 4-6

1. General concepts and techniques
   - Power utility and Epstein Zin utility
   - SDF, pricing equation, risk premium, riskfree rate
   - Campbell Shiller approximation and decompositions
2. Epstein Zin with return decompositions
   - Extended consumption CAPM (BY 2004)
   - Intertemporal CAPM (CV 2004 and CGPT 2018)
   - Without and with stochastic volatility (Break homoskedasticity)
3. Rare disaster
   - Barro (2006, 2009) and Martin (2013) (Break lognormality)

Outlines for Intertemporal Portfolio Choice

1. Static portfolio choice
   - Principle of participation, a small award for risk
   - CARA normal and CRRA lognormal case
2. Myopic portfolio choice
   - Investment rules that do not depend on the investment horizon.
   - CARA normal and CRRA lognormal case
3. Dynamic portfolio choice
   - Hedging interest rates
   - Hedging risk premia

Outlines for Term Structure

1. Refinements on bonds
   - Yield to maturity, forward rate, holding period return, Duration
   - Expectations Hypothesis and Empirical Findings
   - Pricing equation
2. Affine term structure models
   - Completely affine single factor model:
     - Homoskedastic process (Vasicek (1977))→ constant bond risk premium
     - Heteroskedastic process (CIR square-root model)
   - Essentially affine model
     - Homoskedastic and heteroskedastic
     - Single factor and multivariate
   - Key technique: guess and verify & equating coefficient
3. Term structure of risky assets
   - Dividend forwards and strips
   - Housing term structures

Final Notes: Some Techniques

1. Matrix calculation in multi-asset case
2. Approximation, log linearization
3. Normal and log normal distribution
4. Forward and backward iteration (take care of index)
5. Guess and verify to solve for equilibrium
6. Campbell-shiller return decomposition